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A.Sh. Shaldanbayev[1], A.B. Imanbayeva[2], A.Zh. Beisebayeva[3], A.A. Shaldanbayeva[4]

[1]"Silkway" International University, Shymkent;
[2,3]M.Auezov South Kazakhstan State University, Shymkent;
[4]"Regional Social-Innovative University", Shymkent.
shaldanbaev51@mail.ru, aigul_baratovna@bk.ru, akbope_a@mail.ru, altima_a@mail.ru

ON THE SQUARE ROOT OF THE OPERATOR OF STURM-LIOUVILLE FOURTH-ORDER

Abstract. In the present work found the root of the positive operator of the Sturm - Liouville problem of the fourth order, which is invertible composition operator of the Sturm - Liouville problem and its adjoint. The found root does not possess the property of positivity, but is a self-adjoint operator in the essential. One theorem of Putnam of algebraic character is used as a leading idea. It is hoped that the results will find applications in spectral operator theory and theoretical physics.

Keywords. Kato conjecture, dissipative operator, square root of operator, Putnam theorem, deviating argument, fractional powers of an operator, inverse problem, spectrum, unitary operator, self-adjoint operator, positive operator, functional differential operator, spectral theory.

1. Introduction.

Definition. Let H be a Hilbert space. A linear bounded operator B is called positive if $(Bx,x)\geq 0$ for all $x\in H$. We write $B\geq 0$ if B is positive.

Lemma1.1 (On the square root). Let A be a linear bounded operator and $A\geq 0$. Then there is a single operator $B\gg 0$ and $B^2=$ A. Moreover, B commutes with any bounded operator commuting with A. [1. p.219] .

When extending the concept of root to dissipative operators, the hypothesis of Kato [3] arose, consisting in the fact that the domain of the root definition from the operator always coincides with the domain of the root definition from the conjugate operator. But in 1972 A. Macintosh [4] built a counterexample, since then the hypothesis has been slightly reformulated: to find the largest class of operators that satisfies this condition, and is now given very active research in this direction [5-57].

Many operators of theoretical physics have square roots [57–64]; in particular, the square root of an operator in a Banach space was found in [65, pp.169-176]. We give an excerpt from this work.

Consider a generating operator A in a Banach space $\mathcal{B}$ with the following properties:

1) The operator $(I+\gamma^2A)^{-1}$ exists, is defined everywhere in $\mathcal{B}$ and is bounded by one;

2) The operator A^{-1} exists;

3) $\|e^{iAt}\|\leq M,\ -\infty<t<+\infty$.

Under these conditions, the following lemmas are valid.

Lemma 1.1. Operator

$$T=\frac{2e^{-i\pi/4}}{\sqrt{\pi}}A\int_0^\infty e^{iAx^2}dx$$

exists as an operator in B on the area $D(A)$.

Lemma 1.2. For any $y \in D(A)$ the equality is true

$$T^2 g = Ag.$$

In connection with these results, the following problem arises.

Formulation of the problem. Let a reversible Sturm-Liouville operator L

$$Ly = -y''(x), \qquad x \in (0,1)$$

$$\begin{cases} a_{11}y(0) + a_{12}y'(0) + a_{13}y(1) + a_{14}y'(1) = 0, \\ a_{21}y(0) + a_{22}y'(0) + a_{23}y(1) + a_{24}y'(1) = 0, \end{cases}$$

where a_{ij} $(i = 1,2; j = 1,2,3,4) -$ are complex numbers. Then it takes the following form:

$$Ly = -y''(x), \ x \in (0,1) \tag{1}$$

$$\begin{cases} \Delta_{13}y(0) - (\Delta_{12} + \Delta_{32})y'(0) - \Delta_{13}y(1) - \\ -(\Delta_{14} + \Delta_{34})y'(1) = 0; \\ (\Delta_{12} + \Delta_{13} + \Delta_{14})y(0) - (\Delta_{32} + \Delta_{42})y'(0) + \\ +(\Delta_{32} + \Delta_{34})y(1) - (\Delta_{34} + \Delta_{24})y'(1) = 0; \end{cases} \tag{2}$$

conjugate which has the form:

$$L^+ z = -z''(x), \ x \in (0,1), \tag{1$^+$}$$

$$\begin{cases} \bar{\Delta}_{13}z(0) - (\bar{\Delta}_{34} + \bar{\Delta}_{32})z'(0) - \bar{\Delta}_{13}z(1) - \\ -(\bar{\Delta}_{14} + \bar{\Delta}_{12})z'(1) = 0; \\ (\Delta_{12} + \Delta_{13} + \Delta_{14})y(0) - (\Delta_{32} + \Delta_{42})y'(0) + \\ +(\Delta_{32} + \Delta_{34})y(1) - (\Delta_{34} + \Delta_{24})y'(1) = 0. \end{cases} \tag{2$^+$}$$

The question is whether there is a unitary operator.

$$T = i\cos\varphi\, I + \sin\varphi \cdot S,$$

such that the formula takes place

$$TL = L^+ T^*, \tag{3}$$

where $I -$ is an identical operator and S is:

$$Su(x) = u(1 - x). \tag{4}$$

2. Research methods.

As a suggestive idea, we take the following theorem of Putnam.

Theorem [2. p.337]. $M, N, T \in \mathcal{B}(H)$ Suppose that the operators M, N are normal and the operator T is invertible. Suppose that

$$M = TNT^{-1}. \tag{*}$$

If $T = UP -$ the polar decomposition of the operator T, then

$$M = UNU^{-1}.$$

Two operators connected by a relation (*) are called similar. If $U -$is a unitary operator and the relation (1.9) is satisfied, then the operators M and N are called unitary equivalent. Thus, this theorem establishes that such normal operators are unitary equivalent.

Our operators A, B are Hermitian (i.e, symmetric), and such operators belong to the class of normal operators; so there is a unitary operator T such that

$$AT = TB.$$

We believe that this particular operator is the solution of the equations:

$$(Tl)^2 = l^+ l = A, \qquad (lT)^2 = ll^+ = B.$$

Perhaps the operator T we need to impose additional conditions?!

From formula

$$(Tl)^2 = Tl \cdot Tl = l^* l,$$

we see that we need to require $Tl = l^* T^*$, then

$$Tl \cdot Tl = l^* \underbrace{T^* T}_{I} l = l^* l = A \,,$$

Next, from $Tl = l^* T^*$, we have

$$TlT = l^*, \qquad lT = T^{-1} l^* = T^* l^*,$$

then

$$(lT)^2 = lT \cdot lT = |lT = T^* l^*| = lT \cdot T^* l^* = l l^* = B.$$

In addition,

$$TB = Tll^* = l^* T^* l^* = l^* lT = AT.$$

We proved the following Lemma.

Lemma 2.1. If $T -$ is a unitary operator satisfying the condition

$$(Tl)^* = Tl = l^* T^*,$$

then we have the formula

a) $(Tl)^2 = l^* l = A$,

b) $(lT)^2 = l l^* = B$,

c) $AT = TB$.

Thus, the problem was reduced to finding a unitary operator T, with the property, $Tl = l^* T^*$. This Lemma forms the basis of our method.

3. Research Results.

Let$y(x) \in D(L)$, then

$$z(x) = T^* y(x) = -i \cos\varphi \, y(x) + \sin\varphi \cdot y(1-x) \in D(L^+),$$

therefore, there are formulas

$$z(x) = -i \cos\varphi \, y(x) + \sin\varphi \cdot y(1-x),$$
$$z'(x) = -i \cos\varphi \, y'(x) - \sin\varphi \cdot y'(1-x),$$
$$z(0) = -i \cos\varphi \, y(0) + \sin\varphi \cdot y(1),$$
$$z'(0) = -i \cos\varphi \, y'(0) - \sin\varphi \cdot y'(1),$$
$$z(1) = -i \cos\varphi \, y(1) + \sin\varphi \cdot y(1),$$
$$z'(1) = -i \cos\varphi \, y'(1) - \sin\varphi \cdot y'(0),$$
$$z(0) - z(1) = (i \cos\varphi + \sin\varphi) y(1) - (i \cos\varphi + \sin\varphi) y(0) =$$
$$= (i \cos\varphi + \sin\varphi)[y(1) - y(0)],$$
$$\overline{\Delta_{13}}(i \cos\varphi + \sin\varphi)[y(1) - y(0)] + (\overline{\Delta_{32}} + \overline{\Delta_{34}})$$
$$[i \cos\varphi \, y'(0) + \sin\varphi \cdot y'(1)] +$$
$$+(\overline{\Delta_{12}} + \overline{\Delta_{14}})[i \cos\varphi \, y'(1) + \sin\varphi \cdot y'(0)] = 0;$$
$$\overline{\Delta_{13}}(i \cos\varphi + \sin\varphi)[y(1) - y(0)] +$$
$$+[i \cos\varphi \, (\overline{\Delta_{32}} + \overline{\Delta_{34}}) + \sin\varphi \, (\overline{\Delta_{12}} + \overline{\Delta_{14}})] y'(0) +$$
$$+[\sin\varphi \, (\overline{\Delta_{32}} + \overline{\Delta_{34}}) + i \cos\varphi \, (\overline{\Delta_{12}} + \overline{\Delta_{14}})] y'(1) = 0;$$

$$(\overline{\Delta_{12}}+\overline{\Delta_{14}}+\overline{\Delta_{34}})[-i\cos\varphi\, y(0)+\sin\varphi\cdot y(1)]+$$
$$+(\overline{\Delta_{32}}+\overline{\Delta_{34}})[i\cos\varphi\, y'(0)+\sin\varphi\cdot y'(1)]+$$
$$+(\overline{\Delta_{12}}+\overline{\Delta_{32}})[-i\cos\varphi\, y(1)+\sin\varphi\cdot y(0)]+(\overline{\Delta_{12}}+\overline{\Delta_{24}})\cdot$$
$$\cdot[i\cos\varphi\, y'(1)+\sin\varphi\cdot y'(0)]=$$
$$=[-i\cos\varphi\,(\overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}})+(\overline{\Delta_{12}}+\overline{\Delta_{32}})\sin\varphi]y(0)+$$
$$+[\sin\varphi\,(\overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}})-i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{32}})]y(1)+$$
$$+[(\overline{\Delta_{32}}+\overline{\Delta_{42}})\sin\varphi+i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{24}})]y'(1)=0.$$

This boundary condition corresponds to the matrix

$$\begin{pmatrix} -\overline{\Delta_{13}}(\sin\varphi+i\cos\varphi) \\ -i\cos\varphi\,(\overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}})+(\overline{\Delta_{12}}+\overline{\Delta_{32}})\sin\varphi \end{pmatrix.$$
$$i\cos\varphi\,(\overline{\Delta_{32}}+\overline{\Delta_{34}})+\sin\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{14}})$$
$$i\cos\varphi\,(\overline{\Delta_{32}}+\overline{\Delta_{42}})+\sin\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{24}})$$

$$\Delta_{13}(\sin\varphi+i\cos\varphi)$$
$$\sin\varphi\,(\overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}})-i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{32}})$$

$$\left.\begin{matrix}\sin\varphi\,(\overline{\Delta_{32}}+\overline{\Delta_{34}})+i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{14}}) \\ (\overline{\Delta_{32}}+\overline{\Delta_{42}})\sin\varphi+i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{24}})\end{matrix}\right).$$

Combining the obtained system of equations with respect to the unknowns: $y(0), y'(0), y(1), y'(1)$ with the previously known boundary conditions (2), we obtain a system of homogeneous algebraic equations. Since this resulting system of equations obviously has nontrivial solutions, its determinant is zero.

Taking advantage of this fact, we obtain one equation to determine the unknown value φ.

$$\begin{cases} 1)\ \Delta_{13}y(0)-(\Delta_{12}+\Delta_{32})y'(0)-\Delta_{13}y(1)- \\ (\Delta_{14}+\Delta_{34})y'(1)=0; \\ 2)\ (\Delta_{12}+\Delta_{13}+\Delta_{14})y(0)-(\Delta_{32}+\Delta_{42})y'(0)+ \\ (\Delta_{32}+\Delta_{34})y(1)-(\Delta_{34}+\Delta_{24})y'(1)=0; \\ 3)\ \overline{\Delta_{13}}(\sin\varphi+i\cos\varphi)y(0)+[i\cos\varphi\,(\overline{\Delta_{32}}+\overline{\Delta_{34}})+(\overline{\Delta_{12}}+\overline{\Delta_{14}})\sin\varphi]\cdot \\ \cdot y'(0)-\Delta_{13}(\sin\varphi+i\cos\varphi)y(1)+ \\ +[(\overline{\Delta_{32}}+\overline{\Delta_{34}})\sin\varphi+i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{14}})]y'(1)=0; \\ 4)\ [-i\cos\varphi\,(\overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}})+\sin\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{32}})]y(0)+ \\ +[i\cos\varphi\,(\overline{\Delta_{32}}+\overline{\Delta_{42}})+\sin\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{24}})]y'(0)+ \\ [(\overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}})\cdot\sin\varphi-i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{32}})]y(1)+ \\ +[(\overline{\Delta_{32}}+\overline{\Delta_{42}})\sin\varphi+i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{24}})]y'(1)=0. \end{cases}$$

This system of homogeneous algebraic equations is denoted by (5).

The determinant of the system of equations (5) has the following form:

$$\begin{bmatrix} \Delta_{13} \\ \Delta_{12}+\Delta_{13}+\Delta_{14} \\ \overline{\Delta_{13}}(\sin\varphi+i\cos\varphi) \\ -i\cos\varphi\,(\overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}})+\sin\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{32}}) \end{matrix}$$

$$
\left.\begin{matrix}
-(\Delta_{12}+\Delta_{32}) & -\Delta_{13} & -(\Delta_{14}+\Delta_{34}) \\
-(\Delta_{32}+\Delta_{42}) & \Delta_{32}+\Delta_{34} & -(\Delta_{34}+\Delta_{24}) \\
i\cos\varphi\,(\overline{\Delta_{32}}+\overline{\Delta_{34}})+(\overline{\Delta_{12}}+\overline{\Delta_{14}})\sin\varphi & -\overline{\Delta_{13}}(\sin\varphi+i\cos\varphi) & (\overline{\Delta_{32}}+\overline{\Delta_{34}})\sin\varphi+i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{14}}) \\
[i\cos\varphi\,(\overline{\Delta_{32}}+\overline{\Delta_{42}})+\sin\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{24}})] & (\overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}})\sin\varphi-i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{32}}) & (\overline{\Delta_{32}}+\overline{\Delta_{42}})\sin\varphi+i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{24}})
\end{matrix}\right].
$$

This determinant splits into the sum of two determinants:

$$
\begin{bmatrix}
\Delta_{13} & -(\Delta_{12}+\Delta_{32}) & -\Delta_{13} & -(\Delta_{14}+\Delta_{34}) \\
\Delta_{12}+\Delta_{13}+\Delta_{14} & -(\Delta_{32}+\Delta_{42}) & \Delta_{32}+\Delta_{34} & -(\Delta_{34}+\Delta_{24}) \\
-\overline{\Delta_{13}}\sin\varphi & (\overline{\Delta_{12}}+\overline{\Delta_{14}})\sin\varphi & \overline{\Delta_{13}}\sin\varphi & (\overline{\Delta_{32}}+\overline{\Delta_{34}})\sin\varphi \\
(\overline{\Delta_{12}}+\overline{\Delta_{32}})\sin\varphi & (\overline{\Delta_{12}}+\overline{\Delta_{24}})\sin\varphi & (\overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}})\sin\varphi & (\overline{\Delta_{32}}+\overline{\Delta_{42}})\sin\varphi
\end{bmatrix}
+
\begin{bmatrix}
\Delta_{13} & -(\Delta_{12}+\Delta_{32}) & -\Delta_{13} & -(\Delta_{14}+\Delta_{34}) \\
\Delta_{12}+\Delta_{13}+\Delta_{14} & -(\Delta_{32}+\Delta_{42}) & \Delta_{32}+\Delta_{34} & -(\Delta_{34}+\Delta_{24}) \\
-\overline{\Delta_{13}}i\cos\varphi & i\cos\varphi\,(\overline{\Delta_{32}}+\overline{\Delta_{34}}) & i\cdot\overline{\Delta_{13}}\cos\varphi & i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{14}}) \\
-i\cos\varphi\,(\overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}}) & i\cos\varphi\,(\overline{\Delta_{32}}+\overline{\Delta_{42}}) & -i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{32}}) & i\cos\varphi\,(\overline{\Delta_{12}}+\overline{\Delta_{24}})
\end{bmatrix}.
$$

This matrix is denoted by (6).
(a) Calculate the first determinant:

$$
sin^2\varphi\begin{bmatrix}
\Delta_{13} & -(\Delta_{12}+\Delta_{32}) & -\Delta_{13} & -(\Delta_{14}+\Delta_{34}) \\
\Delta_{12}+\Delta_{13}+\Delta_{14} & -(\Delta_{32}+\Delta_{42}) & \Delta_{32}+\Delta_{34} & -(\Delta_{34}+\Delta_{24}) \\
-\overline{\Delta_{13}} & \overline{\Delta_{12}}+\overline{\Delta_{14}} & \overline{\Delta_{13}} & \overline{\Delta_{32}}+\overline{\Delta_{34}} \\
\overline{\Delta_{12}}+\overline{\Delta_{32}} & \overline{\Delta_{12}}+\overline{\Delta_{24}} & \overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}} & \overline{\Delta_{32}}+\overline{\Delta_{42}}
\end{bmatrix}=
$$

$$
=sin^2\varphi\begin{bmatrix}
0 & -\Delta & -\Delta_{13} & -(\Delta_{14}+\Delta_{34}) \\
\Delta & 0 & \Delta_{32}+\Delta_{34} & -(\Delta_{34}+\Delta_{24}) \\
0 & \overline{\Delta} & \overline{\Delta_{13}} & \overline{\Delta_{32}}+\overline{\Delta_{34}} \\
\overline{\Delta} & \overline{\Delta} & \overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}} & \overline{\Delta_{32}}+\overline{\Delta_{42}}
\end{bmatrix}=
$$

$$
=sin^2\varphi\begin{bmatrix}
0 & -\Delta & -\Delta_{13} & -(\Delta_{14}+\Delta_{34}) \\
\Delta & 0 & \Delta_{32}+\Delta_{34} & -(\Delta_{34}+\Delta_{24}) \\
0 & \overline{\Delta} & \overline{\Delta_{13}} & \overline{\Delta_{32}}+\overline{\Delta_{34}} \\
\overline{\Delta} & 0 & \overline{\Delta_{14}}+\overline{\Delta_{34}} & -(\overline{\Delta_{24}}+\overline{\Delta_{34}})
\end{bmatrix}=
$$

$$= sin^2\varphi \left\{ -\Delta \begin{vmatrix} -\Delta & -\Delta_{13} & -(\Delta_{14}+\Delta_{34}) \\ \overline{\Delta} & \overline{\Delta_{13}} & \overline{\Delta_{32}}+\overline{\Delta_{34}} \\ 0 & \overline{\Delta_{14}}+\overline{\Delta_{34}} & -(\overline{\Delta_{24}}+\overline{\Delta_{34}}) \end{vmatrix} - \overline{\Delta} \begin{vmatrix} -\Delta & -\Delta_{13} & -(\Delta_{14}+\Delta_{34}) \\ 0 & \Delta_{32}+\Delta_{34} & -(\Delta_{34}+\Delta_{24}) \\ \overline{\Delta} & \overline{\Delta_{13}} & \overline{\Delta_{32}}+\overline{\Delta_{34}} \end{vmatrix} \right\} =$$

$$= sin^2\varphi \left\{ \Delta \begin{vmatrix} \Delta & \Delta_{13} & \Delta_{14}+\Delta_{34} \\ \overline{\Delta} & \overline{\Delta_{13}} & \overline{\Delta_{32}}+\overline{\Delta_{34}} \\ 0 & \overline{\Delta_{14}}+\overline{\Delta_{34}} & -(\overline{\Delta_{24}}+\overline{\Delta_{34}}) \end{vmatrix} - \overline{\Delta} \begin{vmatrix} \Delta & \Delta_{13} & \Delta_{14}+\Delta_{34} \\ \overline{\Delta} & \overline{\Delta_{13}} & \overline{\Delta_{32}}+\overline{\Delta_{34}} \\ 0 & \Delta_{32}+\Delta_{34} & -(\Delta_{34}+\Delta_{24}) \end{vmatrix} \right\} =$$

$$= sin^2\varphi \left\{ \Delta \cdot \left[-(\overline{\Delta_{14}}+\overline{\Delta_{34}}) \begin{vmatrix} \Delta & \Delta_{14}+\Delta_{34} \\ \overline{\Delta} & \overline{\Delta_{32}}+\overline{\Delta_{34}} \end{vmatrix} - (\overline{\Delta_{24}}+\overline{\Delta_{34}}) \begin{vmatrix} \Delta & \Delta_{13} \\ \overline{\Delta} & \overline{\Delta_{13}} \end{vmatrix} \right] - \right.$$

$$\left. \overline{-\Delta \left[-(\Delta_{32}+\Delta_{34}) \begin{vmatrix} \Delta & \Delta_{14}+\Delta_{34} \\ \overline{\Delta} & \overline{\Delta_{32}}+\overline{\Delta_{34}} \end{vmatrix} - (\Delta_{34}+\Delta_{24}) \begin{vmatrix} \Delta & \Delta_{13} \\ \overline{\Delta} & \overline{\Delta_{13}} \end{vmatrix} \right]} \right\} =$$

$$= sin^2\varphi \cdot \left\{ \begin{vmatrix} \Delta & \Delta_{14}+\Delta_{34} \\ \overline{\Delta} & \overline{\Delta_{32}}+\overline{\Delta_{34}} \end{vmatrix} \cdot [\overline{\Delta}(\Delta_{32}+\Delta_{34}) - \Delta(\overline{\Delta_{14}}+\overline{\Delta_{34}})] + \right.$$

$$\left. + \begin{vmatrix} \Delta & \Delta_{13} \\ \overline{\Delta} & \overline{\Delta_{13}} \end{vmatrix} \cdot [\overline{\Delta}(\Delta_{24}+\Delta_{34}) - \Delta(\overline{\Delta_{24}}+\overline{\Delta_{34}})] \right\} =$$

$$= sin^2\varphi \cdot \{[\Delta(\overline{\Delta_{32}}+\overline{\Delta_{34}}) - \overline{\Delta}(\Delta_{14}+\Delta_{34})] \cdot [\overline{\Delta}(\Delta_{32}+\Delta_{34}) - \Delta(\overline{\Delta_{14}}+\overline{\Delta_{34}})] +$$

$$+(\Delta \cdot \overline{\Delta_{13}} - \overline{\Delta} \cdot \Delta_{13})[\overline{\Delta}(\Delta_{24}+\Delta_{34}) - \Delta(\overline{\Delta_{24}}+\overline{\Delta_{34}})]\} =$$

$$= sin^2\varphi \cdot [|\Delta|^2 \cdot |\Delta_{32}+\Delta_{34}|^2 - \Delta^2(\overline{\Delta_{32}}+\overline{\Delta_{34}})(\overline{\Delta_{14}}+\overline{\Delta_{34}}) - \overline{\Delta}^2(\Delta_{14}+\Delta_{34})(\Delta_{32}+\Delta_{34}) +$$

$$+|\Delta|^2 \cdot |\Delta_{14}+\Delta_{34}|^2 + |\Delta|^2 \cdot \overline{\Delta_{13}}(\Delta_{24}+\Delta_{34}) - \Delta^2\overline{\Delta_{13}}(\overline{\Delta_{24}}+\overline{\Delta_{34}}) -$$

$$-\overline{\Delta}^2\Delta_{13}(\Delta_{24}+\Delta_{34}) + |\Delta|^2\Delta_{13}(\overline{\Delta_{24}}+\overline{\Delta_{34}})] =$$

$$= sin^2\varphi \cdot \{|\Delta|^2 \cdot [|\Delta_{32}+\Delta_{34}|^2 + |\Delta_{14}+\Delta_{34}|^2] -$$

$$-\Delta^2 \cdot [(\overline{\Delta_{32}}+\overline{\Delta_{34}})(\overline{\Delta_{14}}+\overline{\Delta_{34}}) + \overline{\Delta_{13}}(\overline{\Delta_{24}}+\overline{\Delta_{34}})] -$$

$$-\overline{\Delta}^2[(\Delta_{14}+\Delta_{34})(\Delta_{32}+\Delta_{34}) + \Delta_{13}(\Delta_{24}+\Delta_{34})] +$$

$$+|\Delta|^2[\overline{\Delta_{13}}(\Delta_{24}+\Delta_{34}) + \Delta_{13}(\overline{\Delta_{24}}+\overline{\Delta_{34}})]\};$$

Using the obvious formula $\Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{32} = \Delta_{12}\Delta_{34}$ transform the coefficients of the resulting expression

$$(\Delta_{14}+\Delta_{34})(\Delta_{32}+\Delta_{34}) + \Delta_{13}(\Delta_{24}+\Delta_{34}) = \Delta_{14}\Delta_{32} + \Delta_{14}\Delta_{34} + \Delta_{34}\Delta_{32} +$$

$$+\Delta_{34}^2 + \Delta_{13}\Delta_{24} + +\Delta_{13}\Delta_{34} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{34} + \Delta_{34}\Delta_{32} + \Delta_{34}^2 + \Delta_{13}\Delta_{34} =$$

$$= \Delta_{34}(\Delta_{12} + \Delta_{13} + \Delta_{14} + \Delta_{32} + \Delta_{34}) = \Delta_{34} \cdot \Delta.$$

Thus, the value of the first determinant will be

$$sin^2\varphi \cdot \{|\Delta|^2 \cdot [|\Delta_{32}+\Delta_{34}|^2 + |\Delta_{14}+\Delta_{34}|^2] -$$

$$-\Delta^2 \cdot \overline{\Delta_{34} \cdot \Delta} - \overline{\Delta}^2 \cdot \Delta_{34} \cdot \Delta + |\Delta|^2[\overline{\Delta_{13}}(\Delta_{24}+\Delta_{34}) + \Delta_{13}(\overline{\Delta_{24}}+\overline{\Delta_{34}})]\} =$$

$$sin^2\varphi \cdot \{|\Delta|^2 \cdot [|\Delta_{32}+\Delta_{34}|^2 + |\Delta_{14}+\Delta_{34}|^2] - [|\Delta|^2\Delta \cdot \overline{\Delta_{34}} + \overline{\Delta} \cdot \Delta_{34}|\Delta|^2] +$$

$$+|\Delta|^2[\overline{\Delta_{13}}(\Delta_{24}+\Delta_{34})+\Delta_{13}(\overline{\Delta_{24}}+\overline{\Delta_{34}})]\}$$

$$=|\Delta|^2\cdot sin^2\varphi\cdot\{[|\Delta_{32}+\Delta_{34}|^2+|\Delta_{14}+\Delta_{34}|^2]-$$

$$-[\Delta\cdot\overline{\Delta_{34}}+\bar{\bar{\Delta}}\cdot\Delta_{34}]+[\overline{\Delta_{13}}(\Delta_{24}+\Delta_{34})+\Delta_{13}(\overline{\Delta_{24}}+\overline{\Delta_{34}})]\}.$$

Thus, the coefficient at $sin^2\varphi$ turned out to be a real value.
Now we calculate the second determinant from formula (6).
b)

$$-cos^2\varphi\begin{bmatrix}\Delta_{13} & -(\Delta_{12}+\Delta_{32}) & -\Delta_{13} & -(\Delta_{14}+\Delta_{34})\\ \Delta_{12}+\Delta_{13}+\Delta_{14} & -(\Delta_{32}+\Delta_{42}) & \Delta_{32}+\Delta_{34} & -(\Delta_{34}+\Delta_{24})\\ -\overline{\Delta_{13}} & \overline{\Delta_{32}}+\overline{\Delta_{34}} & \overline{\Delta_{13}} & \overline{\Delta_{12}}+\overline{\Delta_{14}}\\ -(\overline{\Delta_{13}}+\overline{\Delta_{14}}+\overline{\Delta_{34}}) & \overline{\Delta_{32}}+\overline{\Delta_{42}} & -(\overline{\Delta_{12}}+\overline{\Delta_{32}}) & \overline{\Delta_{12}}+\overline{\Delta_{24}}\end{bmatrix}=$$

$$=-cos^2\varphi\begin{bmatrix}0 & -\Delta & -\Delta_{13} & -(\Delta_{14}+\Delta_{34})\\ \Delta & 0 & \Delta_{32}+\Delta_{34} & -(\Delta_{34}+\Delta_{24})\\ 0 & \bar{\Delta} & \overline{\Delta_{13}} & \overline{\Delta_{12}}+\overline{\Delta_{14}}\\ -\bar{\Delta} & 0 & -(\overline{\Delta_{12}}+\overline{\Delta_{32}}) & \overline{\Delta_{12}}+\overline{\Delta_{24}}\end{bmatrix}=$$

$$=-cos^2\varphi\left\{-\Delta\begin{vmatrix}-\Delta & -\Delta_{13} & -(\Delta_{14}+\Delta_{34})\\ \bar{\Delta} & \overline{\Delta_{13}} & \overline{\Delta_{12}}+\overline{\Delta_{14}}\\ 0 & -(\overline{\Delta_{12}}+\overline{\Delta_{32}}) & \overline{\Delta_{12}}+\overline{\Delta_{24}}\end{vmatrix}+\bar{\Delta}\begin{vmatrix}-\Delta & -\Delta_{13} & -(\Delta_{14}+\Delta_{34})\\ 0 & \Delta_{32}+\Delta_{34} & -(\Delta_{34}+\Delta_{24})\\ \bar{\Delta} & \overline{\Delta_{13}} & \overline{\Delta_{12}}+\overline{\Delta_{14}}\end{vmatrix}\right\}=$$

$$=-cos^2\varphi\left\{\Delta\cdot\begin{vmatrix}\Delta & \Delta_{13} & \Delta_{14}+\Delta_{34}\\ \bar{\Delta} & \overline{\Delta_{13}} & \overline{\Delta_{12}}+\overline{\Delta_{14}}\\ 0 & -(\overline{\Delta_{12}}+\overline{\Delta_{32}}) & \overline{\Delta_{12}}+\overline{\Delta_{24}}\end{vmatrix}+\bar{\Delta}\cdot\begin{vmatrix}\Delta & \Delta_{13} & \Delta_{14}+\Delta_{34}\\ \bar{\Delta} & \overline{\Delta_{13}} & \overline{\Delta_{12}}+\overline{\Delta_{14}}\\ 0 & \Delta_{32}+\Delta_{34} & -(\Delta_{24}+\Delta_{34})\end{vmatrix}\right\}=$$

$$=-cos^2\varphi\left\{\Delta\cdot\left[(\overline{\Delta_{12}}+\overline{\Delta_{32}})\begin{vmatrix}\Delta & \Delta_{14}+\Delta_{34}\\ \bar{\Delta} & \overline{\Delta_{12}}+\overline{\Delta_{14}}\end{vmatrix}+(\overline{\Delta_{12}}+\overline{\Delta_{24}})\begin{vmatrix}\Delta & \Delta_{13}\\ \bar{\Delta} & \overline{\Delta_{13}}\end{vmatrix}\right]+\right.$$

$$\left.+\bar{\Delta}\left[-(\Delta_{32}+\Delta_{34})\begin{vmatrix}\Delta & \Delta_{14}+\Delta_{34}\\ \bar{\Delta} & \overline{\Delta_{12}}+\overline{\Delta_{14}}\end{vmatrix}-(\Delta_{24}+\Delta_{34})\begin{vmatrix}\Delta & \Delta_{13}\\ \bar{\Delta} & \overline{\Delta_{13}}\end{vmatrix}\right]\right\}=$$

$$=cos^2\varphi\cdot\left\{\begin{vmatrix}\Delta & \Delta_{14}+\Delta_{34}\\ \bar{\Delta} & \overline{\Delta_{12}}+\overline{\Delta_{14}}\end{vmatrix}\cdot[\Delta(\overline{\Delta_{12}}+\overline{\Delta_{32}})-\bar{\Delta}(\Delta_{32}+\Delta_{34})]+\right.$$

$$\left.+\begin{vmatrix}\Delta & \Delta_{13}\\ \bar{\Delta} & \overline{\Delta_{13}}\end{vmatrix}\cdot[\Delta(\overline{\Delta_{12}}+\overline{\Delta_{24}})-\bar{\Delta}(\Delta_{24}+\Delta_{34})]\right\}=$$

$$=cos^2\varphi\cdot\{[\Delta(\overline{\Delta_{12}}+\overline{\Delta_{14}})-\bar{\Delta}(\Delta_{14}+\Delta_{34})]\cdot[\Delta(\overline{\Delta_{12}}+\overline{\Delta_{32}})-\bar{\Delta}(\Delta_{32}+\Delta_{34})]+$$

$$+(\Delta\cdot\overline{\Delta_{13}}-\Delta_{13}\cdot\bar{\Delta})[\Delta(\overline{\Delta_{12}}+\overline{\Delta_{24}})-\bar{\Delta}(\Delta_{24}+\Delta_{34})]\}.$$

Using the obvious formula $\Delta_{13}\Delta_{24}+\Delta_{14}\Delta_{32}=\Delta_{12}\Delta_{34}$, we transform the expression under the curly bracket.

$$\Delta^2\cdot(\overline{\Delta_{12}}+\overline{\Delta_{14}})(\overline{\Delta_{12}}+\overline{\Delta_{32}})-|\Delta|^2(\overline{\Delta_{12}}+\overline{\Delta_{14}})(\Delta_{32}+\Delta_{34})-|\Delta|^2(\Delta_{14}+\Delta_{34})(\overline{\Delta_{12}}+\overline{\Delta_{32}})+$$
$$+\bar{\Delta}^2(\Delta_{14}+\Delta_{34})(\Delta_{32}+\Delta_{34}); \qquad (7)$$

$$\Delta^2\cdot\overline{\Delta_{13}}(\overline{\Delta_{12}}+\overline{\Delta_{24}})-|\Delta|^2\overline{\Delta_{13}}(\Delta_{24}+\Delta_{34})-$$
$$|\Delta|^2\Delta_{13}(\overline{\Delta_{12}}+\overline{\Delta_{24}})+\bar{\Delta}^2\Delta_{13}(\Delta_{24}+\Delta_{34}); \qquad (8)$$

$$(\overline{\Delta_{12}}+\overline{\Delta_{14}})(\overline{\Delta_{12}}+\overline{\Delta_{32}})+\overline{\Delta_{13}}(\overline{\Delta_{12}}+\overline{\Delta_{24}})=\bar{\Delta}_{12}^2+\overline{\Delta_{12}}\cdot\overline{\Delta_{32}}+\overline{\Delta_{14}}\cdot\overline{\Delta_{12}}+$$
$$\overline{\Delta_{14}}\cdot\overline{\Delta_{32}}+\overline{\Delta_{13}}\cdot\overline{\Delta_{12}}+\overline{\Delta_{13}}\cdot\overline{\Delta_{24}}=$$
$$\bar{\Delta}_{12}^2+\overline{\Delta_{12}}\cdot\overline{\Delta_{32}}+\overline{\Delta_{14}}\cdot\overline{\Delta_{12}}+\overline{\Delta_{12}}\cdot\overline{\Delta_{34}}+\overline{\Delta_{13}}\cdot\overline{\Delta_{12}}=\overline{\Delta_{12}}\cdot\bar{\Delta}. \quad (9)$$

Similarly, we have

$$(\Delta_{14}+\Delta_{34})(\Delta_{32}+\Delta_{34})+\Delta_{13}(\Delta_{24}+\Delta_{34})=\Delta_{14}\Delta_{32}+\Delta_{14}\Delta_{34}+$$
$$\Delta_{34}\Delta_{32}+\Delta_{34}^2+\Delta_{13}\Delta_{24}++\Delta_{13}\Delta_{34}=\Delta_{14}\Delta_{34}+\Delta_{34}\Delta_{32}+\Delta_{34}^2+$$
$$+\Delta_{12}\Delta_{34}+\Delta_{13}\Delta_{34}=\Delta_{34}\cdot\Delta. \quad (10)$$

Adding formulas (7), (8) and considering (9) and (10), we have

$$|\Delta|^2\cdot\Delta\cdot\bar{\Delta}_{12}+|\Delta|^2\bar{\Delta}\cdot\Delta_{34}-$$
$$-|\Delta|^2[(\overline{\Delta_{12}}+\overline{\Delta_{14}})(\Delta_{32}+\Delta_{34})+(\Delta_{14}+\Delta_{34})(\overline{\Delta_{12}}+\overline{\Delta_{32}})+$$
$$+\overline{\Delta_{13}}(\Delta_{24}+\Delta_{34})+\Delta_{13}(\overline{\Delta_{12}}+\overline{\Delta_{24}})]. \quad (11)$$

Now convert the expression under the square bracket.

$$(\overline{\Delta_{12}}+\overline{\Delta_{14}})(\Delta_{32}+\Delta_{34})+(\Delta_{14}+\Delta_{34})(\overline{\Delta_{12}}+\overline{\Delta_{32}})+\overline{\Delta_{13}}(\Delta_{24}+\Delta_{34})+\Delta_{13}(\overline{\Delta_{12}}+\overline{\Delta_{24}})=$$
$$=(\bar{\Delta}-\overline{\Delta_{13}}-\overline{\Delta_{32}}-\overline{\Delta_{34}})(\Delta_{32}+\Delta_{34})+(\Delta-\Delta_{12}-\Delta_{32}-\Delta_{13})(\overline{\Delta_{12}}+\overline{\Delta_{32}})+$$
$$+\overline{\Delta_{13}}(\Delta_{24}+\Delta_{34})+\Delta_{13}(\overline{\Delta_{12}}+\overline{\Delta_{24}})=\bar{\Delta}(\Delta_{32}+\Delta_{34})-|\Delta_{32}+\Delta_{34}|^2-\overline{\Delta_{13}}(\Delta_{32}+\Delta_{34})+$$
$$+\Delta(\overline{\Delta_{12}}+\overline{\Delta_{32}})-|\overline{\Delta_{12}}+\overline{\Delta_{32}}|^2-\Delta_{13}(\overline{\Delta_{12}}+\overline{\Delta_{32}})+\overline{\overline{\Delta_{13}}}(\Delta_{24}+\Delta_{34})+\Delta_{13}(\overline{\Delta_{12}}+\overline{\Delta_{24}})=$$
$$-|\Delta_{32}+\Delta_{34}|^2-|\overline{\Delta_{12}}+\overline{\Delta_{32}}|^2+\overline{\overline{\Delta_{13}}}(\Delta_{24}-\Delta_{32})+\Delta_{13}(\overline{\Delta_{24}}-\overline{\Delta_{32}})+\bar{\Delta}(\Delta_{32}+\Delta_{34})+$$
$$\Delta(\overline{\Delta_{12}}+\overline{\Delta_{32}})=\bar{\Delta}\Delta_{32}+\Delta\bar{\Delta}_{32}+\bar{\Delta}\Delta_{34}+\Delta\bar{\Delta}_{12}+$$
$$+\bar{\Delta}_{13}(\Delta_{24}-\Delta_{32})+\Delta_{13}(\overline{\Delta_{24}}-\overline{\Delta_{32}})-|\Delta_{32}+\Delta_{34}|^2-|\overline{\Delta_{12}}+\overline{\Delta_{32}}|^2; \quad (12)$$

Substituting (12) into (1), we obtain

$$|\Delta|^2\cdot\{|\Delta_{32}+\Delta_{34}|^2+|\overline{\Delta_{12}}+\overline{\Delta_{32}}|^2-$$
$$-[\bar{\Delta}\Delta_{32}+\Delta\bar{\Delta}_{32}+\bar{\Delta}_{13}(\Delta_{24}-\Delta_{32})+\Delta_{13}(\overline{\Delta_{24}}-\overline{\Delta_{32}})]\}.$$

Thus, the second determinant has the form:

$$-cos^2\varphi\{|\Delta_{32}+\Delta_{34}|^2+|\overline{\Delta_{12}}+\overline{\Delta_{32}}|^2-$$
$$-[\bar{\Delta}\Delta_{32}+\Delta\bar{\Delta}_{32}+\bar{\Delta}_{13}(\Delta_{24}-\Delta_{32})+\Delta_{13}(\overline{\Delta_{24}}-\overline{\Delta_{32}})]\}|\Delta|^2.$$

Adding this formula to the first determinant, we obtain the equations for determining φ.

$$|\Delta|^2 sin^2\varphi\{[|\Delta_{32}+\Delta_{34}|^2+|\Delta_{14}+\Delta_{34}|^2]-$$
$$-\left[\Delta\cdot\overline{\Delta_{34}}+\bar{\bar{\Delta}}\cdot\Delta_{34}\right]+[\overline{\Delta_{13}}(\Delta_{24}+\Delta_{34})+\Delta_{13}(\overline{\Delta_{24}}+\overline{\Delta_{34}})]\}-|\Delta|^2\cdot cos^2\varphi\{|\Delta_{32}+\Delta_{34}|^2+$$
$$+|\overline{\Delta_{12}}+\overline{\Delta_{32}}|^2-[\bar{\Delta}\Delta_{32}+\Delta\bar{\Delta}_{32}+\bar{\Delta}_{13}(\Delta_{24}-\Delta_{32})+\Delta_{13}(\overline{\Delta_{24}}-\overline{\Delta_{32}})]\}=0.$$

According to our assumption $\Delta\neq 0$, so we can reduce this value, and as a result we have

$$sin^2\varphi-\{|\Delta_{32}+\Delta_{34}|^2+|\overline{\Delta_{12}}+\overline{\Delta_{32}}|^2-$$
$$-[\bar{\Delta}\Delta_{32}+\Delta\bar{\Delta}_{32}+\bar{\Delta}_{13}(\Delta_{24}-\Delta_{32})+\Delta_{13}(\overline{\Delta_{24}}-\overline{\Delta_{32}})]\}\cdot cos^2\varphi=0.$$

Theorem 3.1. If φ is the solution to the equation

$$\left\{[|\Delta_{32}+\Delta_{34}|^2+|\Delta_{14}+\Delta_{34}|^2]-\left[\Delta\cdot\overline{\Delta_{34}}+\bar{\bar{\Delta}}\cdot\Delta_{34}\right]+\right.$$
$$+[\overline{\Delta_{13}}(\Delta_{24}+\Delta_{34})+\Delta_{13}(\overline{\Delta_{24}}+\overline{\Delta_{34}})]\}\cdot sin^2\varphi-$$
$$\{|\Delta_{32}+\Delta_{34}|^2+|\overline{\Delta_{12}}+\overline{\Delta_{32}}|^2+$$
$$+[\bar{\Delta}\Delta_{32}+\Delta\bar{\Delta}_{32}+\bar{\Delta}_{13}(\Delta_{24}-\Delta_{32})+\Delta_{13}(\overline{\Delta_{24}}-\overline{\Delta_{32}})]\}\cdot cos^2\varphi=0, \quad (13)$$

that is the formula

$$TL=L^+T^*,$$

where

$$T = i\cos\varphi\, I + \sin\varphi\, S,\ \ Sy(x) = y(1-x), \qquad (14)$$

and $I -$ is the identity operator.

Theorem 3.2.

If $\varphi -$is the solution of equation (13), then there are formulas

1) $(TL)^2 = L^*L$;

2) $(LT)^2 = LL^*$;

where $L -$ is the Sturm-Liouville operator,

$$T = i\cos\varphi\, I + \sin\varphi\, S,\ \ Sy(x) = y(1-x), \qquad (14)$$

and $I -$is the identity operator.

4. Discussions.

If

a) $\{[|\Delta_{32}+\Delta_{34}|^2 + |\Delta_{14}+\Delta_{34}|^2]\,[\Delta\cdot\overline{\Delta_{34}} + \bar{\bar{\Delta}}\cdot\Delta_{34}] + [\overline{\Delta_{13}}(\Delta_{24}+\Delta_{34}) + \Delta_{13}(\overline{\Delta_{24}}+\overline{\Delta_{34}})]\} = 0,$

$$\{|\Delta_{32}+\Delta_{34}|^2 + +|\overline{\Delta_{12}}+\overline{\Delta_{32}}|^2 -$$
$$-[\bar{\Delta}\Delta_{32} + \Delta\bar{\Delta}_{32} + \bar{\Delta}_{13}(\Delta_{24}-\Delta_{32}) + \Delta_{13}(\overline{\Delta_{24}}-\overline{\Delta_{32}})]\} \neq 0,$$

that $cos\varphi = 0$ and the operator T takes the form $T = S$, where

$Sy(x) = y(1-x)$;

b) $\{[|\Delta_{32}+\Delta_{34}|^2 + |\Delta_{14}+\Delta_{34}|^2]\,[\Delta\cdot\overline{\Delta_{34}} + \bar{\bar{\Delta}}\cdot\Delta_{34}] + [\overline{\Delta_{13}}(\Delta_{24}+\Delta_{34}) + \Delta_{13}(\overline{\Delta_{24}}+\overline{\Delta_{34}})]\} \neq 0$.

$\{|\Delta_{32}+\Delta_{34}|^2 + +|\overline{\Delta_{12}}+\overline{\Delta_{32}}|^2 - [\bar{\Delta}\Delta_{32} + \Delta\bar{\Delta}_{32} + \bar{\Delta}_{13}(\Delta_{24}-\Delta_{32}) + \Delta_{13}(\overline{\Delta_{24}}-\overline{\Delta_{32}})]\} = 0,$ то $\sin\varphi = 0$, and the operator T takes the form $T=il$, where $I -$is the unit operator.

In the first case, the closure of the SL operator is self-adjoint, and in the second case, the operator L is skew-symmetric, therefore its closure is the normal operator.

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А.Ш. Шалданбаев[1], А.Б. Иманбаева[2],
А.Ж. Бейсебаева[3], А.А. Шалданбаева[4]

[1]"Silkway" Халықаралық Университеті, Шымкент;
[2,3]М.Әуезов атындағы Оңтүстік Қазақстан Мемлекеттік Университеті, Шымкент;
[4]Аймақтық әлеуметтік - инновациялық университеті, Шымкент.

ТӨРТІНШІ РЕТТІ ШТУРМ-ЛИУВИЛЛ ОПЕРАТОРЫНЫҢ КВАДРАТ ТҮБІРІ ТУРАЛЫ

Аннотация. Бұл еңбекте Штурм-Лиувиллдің оң төртінші ретті операторынан квадрат түбір табылды. Ол қайтымды Штурм-Лиувилл операторы мен оның сыңарының композициясы немесе көбейтіндісі. Табылған түбір оң емес, бірақ тегі жалқы операторлар санатына жатады. Бағдарлаушы ретінде Путнамның бір алгебралық теоремасы қолданылды. Зерттеу нәтиежелері операторлардың спектрәлді теортясы мен теориялық физикада қолданыс табады деп күтілуде.

Түйін сөздер. Катоның гипотезасы, диссипативті оператор, оператордың квадрат түбірі, Путнамның теоремасы, ауытқыған аргумент, оператордаң бөлшек түбірлері, кері есептер, спектр, жалқы оператор, оң операторлар, функционәл-дифференциал оператор, спектрәлді теория.

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А.Ш. Шалданбаев[1], А.Б. Иманбаева[2],
А.Ж. Бейсебаева[3], А.А. Шалданбаева[4]

[1]Международный университет “Silkway”, Шымкент;
[2,3]Южно-Казахстанский государственный университет имени М.Ауезова, Шымкент;
[4]Региональный социально-инновационный университет, Шымкент.

О КВАДРАТНОМ КОРНЕ ИЗ ОПЕРАТОРА ШТУРМА-ЛИУВИЛЛЯ ЧЕТВЁРТОГО ПОРЯДКА

Аннотация. В настоящей работе найден корень из положительного оператора Штурма - Лиувилля четвертого порядка, являющегося композицией обратимого оператора Штурма - Лиувилля и его сопряженного. Найденный корень не обладает свойством положительности, но является самосопряженным оператором в существенном. В качестве наводящей идеи использована одна теорема Путнама алгебраического характера. Можно надеяться, что результаты работы найдут приложения в спектральной теории операторов и теоретической физике.

Ключевые слова: гипотеза Като, диссипативный оператор, квадратный корень из оператора, теорема Путнама, отклоняющиеся аргумент, дробные степени оператора, обратная задача, спектр, унитарный оператор, самосопряженный оператор, положительный оператор, функционально-дифференциальный оператор, спектральная теория.

Information about authors:

Shaldanbayev A.Sh. – doctor of physico-mathematical Sciences, Professor, head of the center for mathematical modeling «Silkway» International University, Shymkent; http://orcid.org/0000-0002-7577-8402;

Imanbayeva A.B. - M.Auezov South Kazakhstan State University, Shymkent, Candidate of physical and mathematical Sciences, associate Professor; https://orcid.org/0000-0002-8203-7637;

Beisebayeva A.Zh. - M.Auezov South Kazakhstan State University, Shymkent; https://orcid.org/0000-0003-4839-9156;

Shaldanbayeva A.A. - Regional Social-Innovative University, Shymkent; https://orcid.org/0000-0003-2667-3097

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