NEWS
OF THENATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
PHYSICO-MATHEMATICAL SERIES
ISSN 1991-346X **https://doi.org/10.32014/2019.2518-1726.69**
Volume 6, Number 328 (2019), 14 – 24

UDC 517.958:536.2; 517.968.7; 519.62/.64
MRNTI 27.35.45; 27.33.19; 27.41.19

A.T.Assanova[1,2], E.A.Bakirova[1,2,3], Zh.M.Kadirbayeva[1,2,4]

[1]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;
[2]Institute of Information and Computational Technologies, Almaty, Kazakhstan;
[3]Kazakh National Women's Teacher Training University, Almaty, Kazakhstan;
[4]International Information Technology University, Almaty, Kazakhstan
assanova@math.kz,bakirova1974@mail.ru,zh.kadirbayeva@gmail.com

NUMERICALLY APPROXIMATE METHOD FOR SOLVING OF A CONTROL PROBLEM FOR INTEGRO-DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE

Abstract. A linear boundary value problem with a parameter for integro-differential equations of parabolic type is investigated. Using the spatial variable discretization, the considering problem is approximated by a linear boundary value problem with a parameter for a system of ordinary integro-differential equations.The parameterization method is used for solving the obtained problem. The approximating problem is reduced to an equivalent problem consisting of a special Cauchy problem for the system of Fredholm integro-differential equations, boundary conditions, and continuity conditions of the solution at the partition points. The solution of the Cauchy problem for the system of ordinary differential equations with parameters is constructed using the fundamental matrix of the differential equation. The system of a linear algebraic equations with respect to the parameters are composed by substituting the values of the corresponding points in the boundary condition and the continuity conditions. Numerical method for solving of the problem is suggested, which based on the solving of the constructed system and method of Runge-Kutta 4-th order for solving of the Cauchy problem on the subintervals.

Key words:partial integro-differential equations of parabolic type, problem with parameter, approximation, numericallyapproximate method, algorithm.

Funding: The work was supported by grants (№№ AP05132486, AP05131220, AP05132455) of Ministry of Education and Science of the Republic of Kazakhstan (2018-2020).

Control problems, which are also called boundary value problems with parameters and the problem of parameter identification for a system of ordinary differential and integro-differential equations with parameters, have been actively investigated in recent decades. Models describing reaction-diffusion processes lead to control problems for integro-differential equations of parabolic type [1-17].Questions of existence, uniqueness and stability of solving problems with parameters are very important for development of numerical methods of identification of parameters of the mathematical models described by integro-differential equations of parabolic type [1-17].

In the present paper, linear problem with a parameter for an integro-differential equation of parabolic type is investigated. By discretizing a spatial variable, the considering problem is approximated by a two-point boundary value problem with parameters for a system of Fredholm integro-differential equations with a degenerate kernel.By introducing additional parameters [18-23] as the values of the desired solution at some points of the interval $[0, T]$, where the problem is considered, the obtained problem is reduced to the equivalent problem consisting of a special Cauchy problem for the system of Fredholm integro-differential equations, boundary conditions, and continuity conditions for the solution at the points of partition.Using the integral equation, that equivalent to the special Cauchy problem for the system of Fredholm integro-differential equation and the property of the degeneracy of kernel of the integral term, we obtained a representation of the solution of the special Cauchy problem using the entered parameters at the assumption of invertibility of a some matrix. Based on this representation, a system of algebraic equations with respect to the parameters is constructed from the boundary conditions and the continuity conditions of the solution. We offer algorithm for solving the linearboundaryvalue problem for the equation with degeneratekernel,anditsnumerical implementation.

We consider a linear boundary value problem with a parameter for an integro-differential equation of parabolic type

$$\frac{\partial u}{\partial t} = a(x,t)\frac{\partial^2 u}{\partial x^2} + c(x,t)u + b(x,t)\mu(x) +$$

$$+\varphi(x,t)\int_0^T \psi(x,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega = (0,\omega) \times (0,T), \tag{1}$$

$$u(x,0) = 0, \ x \in [0,\omega], \tag{2}$$

$$u(x,T) = 0, \ x \in [0,\omega], \tag{3}$$

$$u(0,t) = \tilde{\psi}_1(t), \ u(\omega,t) = \tilde{\psi}_2(t), \ t \in [0,T], \tag{4}$$

where $u(x,t)$issought function, $\mu(x)$isunknown functional parameter, functions$a(x,t) \geq a_0 > 0$, $c(x,t) \leq 0$, $b(x,t)$, $\varphi(x,t)$,$\psi(x,t)$, $f(x,t)$are continuous in $t$and Holder continuous in$x$ onΩ; functions$\tilde{\psi}_1(t)$, $\tilde{\psi}_2(t)$are continuous on $[0,T]$. It is assumed that the boundary functions are sufficiently smooth and satisfy the matching conditions.

The solution of the boundary problem (1)-(4) is a pair of functions$(u^*(x,t),\mu^*(x))$,where function $u^*(x,t)$ is continuousonΩ, that has continuouspartial derivatives with respect to x of first order, with respect to t of second order, satisfies the integro-differential equation (1) at $\mu(x) = \mu^*(x)$, $x \in [0,\omega]$,and boundary conditions (2)-(4).

In view of condition (2)-(4), from (1) we obtain two groups of equations for determining $\mu(0)$and$\mu(\omega)$:

$$b(0,0)\mu(0) = \dot{\tilde{\psi}}_1(0) - \varphi(0,0)\int_0^T \psi(0,s)\tilde{\psi}_1(s)ds - f(0,0),$$

$$b(\omega,0)\mu(\omega) = \dot{\tilde{\psi}}_2(0) - \varphi(\omega,0)\int_0^T \psi(\omega,s)\tilde{\psi}_2(s)ds - f(\omega,0),$$

$$b(0,T)\mu(0) = \dot{\tilde{\psi}}_1(T) - \varphi(0,T)\int_0^T \psi(0,s)\tilde{\psi}_1(s)ds - f(0,T),$$

$$b(\omega,T)\mu(\omega) = \dot{\tilde{\psi}}_2(T) - \varphi(\omega,T)\int_0^T \psi(\omega,s)\tilde{\psi}_2(s)ds - f(\omega,T).$$

These relations also are the matching conditions with respect to initial data.

We take $\forall h > 0$ and produce a discretization by x: $x_i = ih, i = \overline{0,P}, Ph = \omega$.

We introduce the notations$u_i(t) = u(ih,t), \mu_i = \mu(ih), a_i(t) = a(ih,t), c_i(t) = c(ih,t), b_i(t) = b(ih,t), \varphi_i(t) = \varphi(ih,t), \psi_i(t) = \psi(ih,t), f_i(t) = f(ih,t), i = \overline{0,P}$.

Problem (1) - (4) is replaced by the following linear boundary value problem with a parameter for an integro-differential equation

$$\frac{du_i}{dt} = a_i(t)\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + c_i(t)u_i + b_i(t)\mu_i +$$

$$+\varphi_i(t)\int_0^T \psi_i(s)u_i(s)ds + f_i(t), \quad i = \overline{1,P-1}, \tag{5}$$

$$u_i(0) = 0, \ i = \overline{0,P}, \tag{6}$$

$$u_i(T) = 0, \ i = \overline{0,P}, \tag{7}$$

$$u_0(t) = \tilde{\psi}_1(t), \ u_P(t) = \tilde{\psi}_2(t), \ t \in [0,T]. \tag{8}$$

The functions$u_0(t)$, $u_P(t)$, and parameters μ_0,μ_P are known.

Problem (5)-(8) will be rewritten in vector-matrix form

$$\frac{du}{dt} = A(t)u + B(t)\mu + \Phi(t)\int_0^T \Psi(s)u(s)ds + F(t), \quad u,\mu \in R^{P-1}, \quad t \in (0,T), \tag{9}$$

$$u(0) = 0, \tag{10}$$

$$u(T) = 0, \tag{11}$$

where$u\,(t) = (u_1(t), u_2(t), \dots, u_{P-1}(t))$, $\mu = (\mu_1, \mu_2, \dots, \mu_{P-1})$-unknown function and parameter,

$$A(t) = \begin{pmatrix} -\frac{2a_1(t)}{h^2} + c_1(t) & \frac{a_1(t)}{h^2} & 0 & \cdots & 0 \\ \frac{a_2(t)}{h^2} & -\frac{2a_2(t)}{h^2} + c_2(t) & \frac{a_2(t)}{h^2} & \cdots & 0 \\ 0 & \frac{a_3(t)}{h^2} & -\frac{2a_3(t)}{h^2} + c_3(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\frac{2a_{P-1}(t)}{h^2} + c_{P-1}(t) \end{pmatrix},$$

$B(t) = diag\{b_1(t), b_2(t), \dots, b_{P-1}(t)\}$,

$\Phi(t) = diag\{\varphi_1(t), \varphi_2(t), \dots, \varphi_{P-1}(t)\}$,

$\Psi(s) = diag\{\psi_1(s), \psi_2(s), \dots, \psi_{P-1}(s)\}$,

$F(t) = \left(\frac{a_1(t)}{h^2}\tilde{\psi}_1(t) + f_1(t), f_2(t), \dots, \frac{a_{P-1}(t)}{h^2}\tilde{\psi}_2(t) + f_{P-1}(t)\right)^{/}$.

Here $(P-1)\times(P-1)$-matricesA(t), B(t), $\Phi(t)$, $\Psi(s)$and$(P-1)$-vector$F(t)$are continuous on$[0,T]$.

The solution to problem (9) - (11) is a pair$(u^*(t), \mu^*)$, where continuous on $[0,T]$and continuously differentiable on $(0,T)$a function $u^*(t)$satisfies the integro-differential equation (9) at $\mu = \mu^*$and conditions (10), (11).

To solve the problem with parameter (9)-(11), the approach developed in [24-26] is used, based on the algorithms of the parameterization method and numerical methods for solving Cauchy problems.

Scheme of the method.Points$0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$ are taken and the interval$[0,T)$ is divided into Nsubintervals:$[0,T) = \cup_{r=1}^{N}[t_{r-1}, t_r)$, which is denoted by$\Delta_N$[20]. The restriction of the function $u(t)$ to the r −th interval $[t_{r-1}, t_r)$ is denoted by $x_r(t)$, i.e. $u_r(t) = u(t)$for $t \in [t_{r-1}, t_r)$, r=$\overline{1,N}$.

Let$C([0,T], R^{P-1})$be the space of continuous on $[0,T]$functions$u: [0,T] \to R^{P-1}$with norm$\|u\|_1 = \max_{t\in[0,T]}\|u(t)\|$; $C([0,T], \Delta_N, R^{(P-1)N})$ - the space of systems of functions$u[t] = (u_1(t), u_2(t), \dots, u_N(t))$, where $u_r: [t_{r-1}, t_r) \to R^{P-1}$ are continuous on$[t_{r-1}, t_r)$ and have finite left-sided limits$\lim_{t\to t_r-0} u_r(t)$ for allr=$\overline{1,N}$, with norm$\|u[\cdot]\|_2 = \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)}\|u_r(t)\|$.

We introduce additional parameters $\lambda_r = u_r(t_{r-1})$, r=$\overline{2,N}$, $\lambda_1 = \mu$.Making the substitution $u_1(t) = z_1(t)$, $u_r(t) = z_r(t) + \lambda_r$on every r-th interval $[t_{r-1}, t_r)$, r=$\overline{2,N}$,we obtain multipoint boundary value problem with parameters

$$\frac{dz_1}{dt} = A(t)z_1 + B(t)\lambda_1 + \Phi(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)z_j(s)ds +$$

$$+\Phi(t)\sum_{j=2}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)\lambda_j ds + F(t), \quad t \in [t_0, t_1), \tag{12}$$

$$z_1(t_0) = 0, \tag{13}$$

$$\frac{dz_r}{dt} = A(t)(z_r + \lambda_r) + B(t)\lambda_1 + \Phi(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)z_j(s)ds +$$

$$+\Phi(t)\sum_{j=2}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)\lambda_j ds + F(t), \quad t \in [t_{r-1}, t_r), \tag{14}$$

$$z_r(t_{r-1}) = 0, \ r=\overline{2,N}, \tag{15}$$

$$\lambda_N + \lim_{t\to T-0} z_N(t) = 0, \tag{16}$$

$$\lim_{t\to t_1-0} z_1(t) = \lambda_2, \tag{17}$$

$$\lambda_s + \lim_{t\to t_s-0} z_s(t) = \lambda_{s+1}, \quad s = \overline{2, N-1}. \tag{18}$$

The solution of the problem with parameters (12)-(18) is a pair $(z^*[t], \lambda^*)$where the function$z^*[t] = (z_1^*(t), z_2^*(t), \dots, z_N^*(t)) \in C([0,T], \Delta_N, R^{(P-1)N})$with continuously differentiable components $z_r^*(t)$on$[t_{r-1}, t_r)$and $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_N^*) \in R^{(P-1)N}$, satisfies the integro-differential equation with parameters (12), (14), initial conditions (13), (15), relations (16)-(18) at$\lambda_j = \lambda_j^*, j = \overline{2,N}$.

If the pair$(u^*(t), \mu^*)$ is a solution of problem (9)-(11), then the pair $(z^*[t], \lambda^*)$with elements $z^*[t] = (z_1^*(t), z_2^*(t), \dots, z_N^*(t)) \in C([0,T], \Delta_N, R^{(P-1)N})$, $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_N^*) \in R^{(P-1)N}$, where $\lambda_1^* = \mu^* \in R^{P-1}$, $z_1^*(t) = u_1^*(t)$, $t \in [t_0, t_1)$, $\lambda_r^* = u_r^*(t_{r-1})$, $z_r^*(t) = u_r^*(t) + u_r^*(t_{r-1})$, $t \in [t_{r-1}, t_r)$, $r = \overline{2,N}$, is the solution of problem (12)-(18). Conversely, if a pair $(\tilde{z}[t], \tilde{\lambda})$with elements $\tilde{z}[t] = (\tilde{z}_1(t), \tilde{z}_2(t), \dots, \tilde{z}_N(t)) \in C([0,T], \Delta_N, R^{(P-1)N})$, $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_N) \in R^{(P-1)N}$, is a solution of (12)-(18) , then the pair $(\tilde{u}(t), \tilde{\mu})$defined by the equalities $\tilde{u}(t) = \tilde{z}_1(t)$, $t \in [t_0, t_1)$,$\tilde{u}(t) = \tilde{z}_r(t) + \tilde{\lambda}_r$,$t \in [t_{r-1}, t_r)$, $r = \overline{2,N}$,$\tilde{u}(T) = \lim_{t\to T-0} \tilde{z}_N(t) + \tilde{\lambda}_N$ and $\tilde{\mu} = \tilde{\lambda}_1$, will be the solution of the original boundary value problem with parameter (9) - (11).

Using the fundamental matrix $X_r(t)$of the differential equation$\frac{dx}{dt} = A(t)x$,$t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, we reduce the solution of a special Cauchy problem for an integro-differential equation with parameters (12)-(15) to an equivalent system of integral equation

$$z_1(t) = X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)\Phi(\tau)\left\{\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)z_j(s)ds + \sum_{j=2}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)\lambda_j ds\right\}d\tau +$$

$$+X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)B(\tau)d\tau\,\lambda_1 + X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)F(\tau)d\tau, t \in [t_0, t_1), \tag{19}$$

$$z_r(t) = X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)\Phi(\tau)\left\{\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)z_j(s)ds + \sum_{j=2}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)\lambda_j ds\right\}d\tau +$$

$$+X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)A(\tau)d\tau\,\lambda_r + X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)B(\tau)d\tau\,\lambda_1 +$$

$$+X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)F(\tau)d\tau, t \in [t_{r-1}, t_r), \ r = \overline{2,N}. \tag{20}$$

Let $\xi = \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)z_j(s)ds$and rewrite the system of integral equations (19), (20) in the form

$$z_1(t) = X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)\Phi(\tau)\left\{\xi + \sum_{j=2}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)\lambda_j ds\right\}d\tau +$$

$$+X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)B(\tau)d\tau\,\lambda_1 + X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)F(\tau)d\tau, t \in [t_0, t_1), \tag{21}$$

$$z_r(t) = X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau)\Phi(\tau) \left\{ \xi + \sum_{j=2}^{N} \int_{t_{j-1}}^{t_j} \Psi(s)\lambda_j ds \right\} d\tau +$$

$$+X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau)A(\tau)d\tau\, \lambda_r + X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau)B(\tau)d\tau\, \lambda_1 +$$

$$+X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau)F(\tau)d\tau\,, t \in [t_{r-1}, t_r),\ r = \overline{2, N}. \quad (22)$$

Multiplying both parts of (21), (22)by $\Psi(t)$, integrating on$[t_{r-1}, t_r]$, and summing by$r$, we obtain a system of linear algebraic equations with respect to$\xi \in R^{P-1}$

$$\xi = G(\Delta_N)\xi + \sum_{r=1}^{N} V_r(\Delta_N)\, \lambda_r + \mathrm{g}(\mathrm{F}, \Delta_N), \quad (23)$$

with $(P-1) \times (P-1)$-matrices

$$G(\Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \Psi(\tau)X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(s)\Phi(s)dsd\tau,$$

$$V_1(\Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \Psi(\tau)X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(s)\mathrm{B}(s)dsd\tau,$$

$$V_r(\Delta_N) = \int_{t_{r-1}}^{t_r} \Psi(\tau)X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(s)A(s)dsd\tau +$$

$$+\sum_{r=1}^{N} \sum_{j=1}^{N} \int_{t_{r-1}}^{t_r} \Psi(\tau)X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(\tau_1)\Phi(\tau_1)d\tau_1 d\tau \int_{t_{j-1}}^{t_j} \Psi(s)ds\,,\ r = \overline{2, N}.$$

and$(P-1)$–vectors

$$\mathrm{g}(\mathrm{F}, \Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \Psi(\tau)X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(s)\mathrm{F}(s)dsd\tau.$$

We write the system (23) in the form

$$[I - G(\Delta_N)]\xi = \sum_{r=1}^{N} V_r(\Delta_N)\, \lambda_r + \mathrm{g}(\mathrm{F}, \Delta_N), \quad (24)$$

where$I$is the identity matrix of $(P-1)$dimension.

The special Cauchy problem (12) - (15) is equivalent to the system of integral equations (19) - (20). This system, due to the degeneracy of the kernel, will be equivalent to the system of algebraic equations (23) with respect to$\xi \in R^{P-1}$. The unique solvability of the special Cauchy problem was investigated in [19, 20]. It has been established that with a sufficiently small step$h > 0 \colon Nh = T$ partitioning a segment$[0, T]$ the special Cauchy problem will be unique solvable.

Let the matrix $I - G(\Delta_N)$be invertible, i.e. exists $[I - G(\Delta_N)]^{-1}$. Then, according to (24), the vector $\xi \in R^{P-1}$is determined by the equality

$$\xi = [I - G(\Delta_N)]^{-1} \sum_{r=1}^{N} V_r(\Delta_N)\, \lambda_r + [I - G(\Delta_N)]^{-1} \mathrm{g}(\mathrm{F}, \Delta_N). \quad (25)$$

In (21), (22), instead of ξ substituting the right-hand side of (25), we obtain the representation of the function $z_r(t)$in terms of$\lambda_r,\ r = \overline{1, N+1}$:

$$z_1(t)=\sum_{j=2}^{N}\left\{X_1(t)\int_{t_0}^{t}X_1^{-1}(\tau)\Phi(\tau)d\tau\left[[I-G(\Delta_N)]^{-1}V_j(\Delta_N)+\int_{t_{j-1}}^{t_j}\Psi(s)ds\right]\right\}\lambda_j+$$

$$+X_1(t)\int_{t_0}^{t}X_1^{-1}(\tau)[\Phi(\tau)[I-G(\Delta_N)]^{-1}V_1(\Delta_N)+B(\tau)]d\tau\,\lambda_1+$$

$$+X_1(t)\int_{t_0}^{t}X_1^{-1}(\tau)[\Phi(\tau)[I-G(\Delta_N)]^{-1}\mathrm{g}(\mathrm{F},\Delta_N)+F(\tau)]d\tau\,,t\in[t_0,t_1), \qquad (26)$$

$$z_r(t)=\sum_{j=2}^{N}\left\{X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)\Phi(\tau)d\tau\left[[I-G(\Delta_N)]^{-1}V_j(\Delta_N)+\int_{t_{j-1}}^{t_j}\Psi(s)ds\right]\right\}\lambda_j+$$

$$+X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)A(\tau)d\tau\,\lambda_r+X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)[\Phi(\tau)[I-G(\Delta_N)]^{-1}V_1(\Delta_N)+B(\tau)]d\tau\,\lambda_1+$$

$$+X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)[\Phi(\tau)[I-G(\Delta_N)]^{-1}\mathrm{g}(\mathrm{F},\Delta_N)+F(\tau)]d\tau\,,t\in[t_{r-1},t_r),\ \ r=\overline{2,N}. \qquad (27)$$

We introduce the notations

$$D_{r,j}(\Delta_N)=X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)\Phi(\tau)d\tau\left[[I-G(\Delta_N)]^{-1}V_j(\Delta_N)+\int_{t_{j-1}}^{t_j}\Psi(s)ds\right],\ \ r\neq j,\qquad \mathrm{r,j}=\overline{2,N},$$

$$D_{r,r}(\Delta_N)=X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)\Phi(\tau)d\tau\left[[I-G(\Delta_N)]^{-1}V_r(\Delta_N)+\int_{t_{j-1}}^{t_j}\Psi(s)ds\right]$$

$$+\,+X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)A(\tau)d\tau\,,\mathrm{r}=\overline{2,N},$$

$$D_{r,1}(\Delta_N)=X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)\Phi(\tau)d\tau\,[I-G(\Delta_N)]^{-1}V_1(\Delta_N)+X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)B(\tau)d\tau\,,\mathrm{r}=\overline{1,N},$$

$$F_r(\Delta_N)=X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)\Phi(\tau)[I-G(\Delta_N)]^{-1}\mathrm{g}(\mathrm{F},\Delta_N)d\tau+X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)F(\tau)d\tau\,,\mathrm{r}=\overline{1,N}.$$

Then from (26), (27) we obtain

$$\lim_{t\to t_r-0}z_r(t)=\sum_{j=1}^{N}D_{r,j}(\Delta_N)\,\lambda_\mathrm{j}+F_r(\Delta_N),\quad \mathrm{r}=\overline{1,N}. \qquad (28)$$

Substituting the corresponding right-hand sides of (28) into the conditions (16) – (18), we obtain a system of linear algebraic equations with respect to the parameters$\lambda_r,\ r=\overline{1,N+1}$:

$$\left[I+D_{N,N}(\Delta_N)\right]\lambda_N+\sum_{j=1}^{N-1}D_{N,j}(\Delta_N)\,\lambda_\mathrm{j}=-F_N(\Delta_N), \qquad (29)$$

$$\sum_{\substack{j=1\\ j\neq 2}}^{N}D_{1,j}(\Delta_N)\,\lambda_\mathrm{j}-\left[I-D_{1,2}(\Delta_N)\right]\lambda_2=-F_1(\Delta_N), \qquad (30)$$

$$\left[I + D_{s,s}(\Delta_N)\right]\lambda_s - \left[I - D_{s,s+1}(\Delta_N)\right]\lambda_{s+1} +$$
$$+\sum_{\substack{j=1 \\ j\neq s, j\neq s+1}}^{N} D_{s,j}(\Delta_N)\lambda_j = -F_s(\Delta_N), \quad s = \overline{2, N-1}. \tag{31}$$

We denote the matrix corresponding to the left side of the system of equations (29) - (31) by $Q_*(\Delta_N)$and write the system in the form

$$Q_*(\Delta_N)\lambda = -F_*(\Delta_N), \ \ \lambda \in R^{(P-1)N}, \tag{32}$$

where $F_*(\Delta_N) = (F_N(\Delta_N), F_1(\Delta_N), \dots, F_{N-1}(\Delta_N)) \in R^{(P-1)N}$.

Cauchy problems for ordinary differential equations on subintervals

$$\frac{dx}{dt} = A(t)x + P(t), \ \ x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \ \ \mathrm{r} = \overline{1, N} \tag{33}$$

are a significant part of proposed algorithm. Here $P(t)$ is either $(n \times n)$ matrix, or n vector, both continuous on $[t_{r-1}, t_r]$, $\mathrm{r} = \overline{1, N}$. Consequently, solution to problem (33) is a square matrix or a vector of dimension n.

Denote by $a(P, t)$ the solution to the Cauchy problem (33). Obviously,

$$a(P, t) = X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau)\mathrm{P}(\tau)d\tau, \ \ t \in [t_{r-1}, t_r],$$

where $X_r(t)$ is a fundamental matrix of differential equation (33) on the r-th interval.

We offer the following numerical implementation of algorithm based on the Runge – Kutta method of 4^{th} order and Simpson's method.

1. Suppose we have a partition $\Delta_N: 0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$. Divide each r-th interval $[t_{r-1}, t_r]$, $\mathrm{r} = \overline{1, N}$, into N_r parts with step $h_r = (t_r - t_{r-1})/N_r$. Assume on each interval $[t_{r-1}, t_r]$ the variable $\hat{t}$ takes its discrete values: $\hat{t} = t_{r-1}, \hat{t} = t_{r-1} + h_r, \dots, \ \hat{t} = t_{r-1} + (N_r - 1)h_r, \hat{t} = t_r$, and denote by $\{t_{r-1}, t_r\}$ the set of such points.

2. Using the Runge Kutta method of 4^{th} order, we find the numerical solutions to Cauchy problems

$$\frac{dx}{dt} = A(t)x + \Phi(t), \ \ x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r],$$

and define the values of $(n \times n)$ matrices $a^{h_r}(\Phi, \hat{t})$on the set$\{t_{r-1}, t_r\}$, $\mathrm{r} = \overline{1, N}$.

3. Using the values of $(n \times n)$ matrices $\Psi(s)$and $a^{h_r}(\Phi, \hat{t})$on $\{t_{r-1}, t_r\}$, and Simpson's method, we calculate the$(n \times n)$matrices

$$\widehat{\Psi}_r^{h_r}(\Phi) = \int_{t_{r-1}}^{t_r} \Psi(\tau)a^{h_r}(\Phi, \tau)d\tau, \ \ \mathrm{r} = \overline{1, N}.$$

Summing up the matrices $\widehat{\Psi}_r^{h_r}(\Phi)$ over r, we find the $(n \times n)$ matrices $G^{\tilde{h}}(\Delta_N) = \sum_{j=1}^{N} \widehat{\Psi}_j^{h_j}(\Phi)$, where $\tilde{h} = (h_1, h_2, \dots, h_N) \in R^P$.

4. Solving the Cauchy problems

$$\frac{dx}{dt} = A(t)x + A(t), \ \ x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r],$$
$$\frac{dx}{dt} = A(t)x + B(t), \ \ x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r],$$
$$\frac{dx}{dt} = A(t)x + F(t), \ \ x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \qquad \mathrm{r} = \overline{1, N},$$

by using again the Runge–Kutta method of 4^{th} order, we find the values of $(n \times n)$ matrices $a(\mathrm{A}, \hat{t}), a(B, \hat{t})$ and n vector $a(F, \hat{t})$ on $\{t_{r-1}, t_r\}$, $\mathrm{r} = \overline{1, N}$.

5. Applying Simpson's method on the set $\{t_{r-1}, t_r\}$, we evaluate the definite integrals

$$\widehat{\Psi}_r^{h_r} = \int_{t_{r-1}}^{t_r} \Psi(\tau)d\tau, \qquad \widehat{\Psi}_r^{h_r}(\mathrm{A}) = \int_{t_{r-1}}^{t_r} \Psi(\tau)a^{h_r}(\mathrm{A}, \tau)d\tau,$$

$$\widehat{\Psi}_r^{h_r}(\mathrm{B}) = \int_{t_{r-1}}^{t_r} \Psi(\tau) a^{h_r}(\mathrm{B},\tau) d\tau, \qquad \widehat{\Psi}_r^{h_r}(F) = \int_{t_{r-1}}^{t_r} \Psi(\tau) a^{h_r}(F,\tau) d\tau, \ \mathrm{r} = \overline{1,N}.$$

By the equalities

$$V_1^{\widetilde{h}}(\Delta_N) = \sum_{j=1}^{N} \widehat{\Psi}_j^{h_j}(\mathrm{B}),$$

$$V_r^{\widetilde{h}}(\Delta_N) = \widehat{\Psi}_r^{h_r}(\mathrm{A}) + \sum_{j=1}^{N} \widehat{\Psi}_j^{h_j}(\Phi) \cdot \widehat{\Psi}_r^{h_r}, \ \mathrm{r} = \overline{2,N},$$

$$g^{\widetilde{h}}(F,\Delta_N) = \sum_{j=1}^{N} \widehat{\Psi}_j^{h_j}(\mathrm{F}).$$

we define the $(n \times n)$ matrices $V_r^{\widetilde{h}}(\Delta_N)$ and n vectors $g^{\widetilde{h}}(F,\Delta_N)$, $\mathrm{r} = \overline{1,N}$.

6. Construct the system of linear algebraic equations with respect to parameters

$Q_*^{\widetilde{h}}(\Delta_N)\lambda = -F_*^{\widetilde{h}}(\Delta_N), \ \lambda \in R^{(P-1)N}$,(34)

Solving the system (34), we find $\lambda^{\widetilde{h}}$. As noted above, the elements of $\lambda^{\widetilde{h}}$=$(\lambda_1^{\widetilde{h}}, \lambda_2^{\widetilde{h}}, \dots, \lambda_N^{\widetilde{h}})$ are the values of approximate solution to problem (12)-(18)at the left-end points of subintervals.

7. To define the values of approximate solution at the remaining points of set $\{t_{r-1}, t_r\}$, we first find

$$\xi^{\widetilde{h}} = \left[I - G^{\widetilde{h}}(\Delta_N)\right]^{-1} \sum_{r=1}^{N} V_r^{\widetilde{h}}(\Delta_N)\, \lambda_r^{\widetilde{h}} + \left[I - G^{\widetilde{h}}(\Delta_N)\right]^{-1} g^{\widetilde{h}}(\mathrm{F},\Delta_N).$$

and then solve the Cauchy problems

$$\frac{du}{dt} = A(t)u + \Phi(t)\left(\xi^{\widetilde{h}} + \sum_{j=2}^{N} \widehat{\Psi}_j^{h_j} \cdot \lambda_j^{\widetilde{h}}\right) + B(t)\lambda_1^{\widetilde{h}} + F(t),$$

$$u(t_0) = 0, \quad t \in [t_0, t_1],$$

$$\frac{du}{dt} = A(t)u + \Phi(t)\left(\xi^{\widetilde{h}} + \sum_{j=2}^{N} \widehat{\Psi}_j^{h_j} \cdot \lambda_j^{\widetilde{h}}\right) + B(t)\lambda_1^{\widetilde{h}} + F(t),$$

$$u(t_{r-1}) = \lambda_r^{\widetilde{h}}, \quad t \in [t_{r-1}, t_r], \ \mathrm{r} = \overline{2,N}.$$

And the solutions to Cauchy problems are found by the Runge–Kutta method of 4th order. Thus,the algorithm allows us to find the numerical solution to the problem (9)-(11).

So, we propose the numerically approximate method for solving of the original problem (1)-(4).

Example. We consider a linear boundary value problem with a parameter for an integro-differential equation of parabolic type

$$\frac{\partial u}{\partial t} = a(x,t)\frac{\partial^2 u}{\partial x^2} + c(x,t)u + b(x,t)\mu(x) +$$

$$+\varphi(x,t)\int_0^T \psi(x,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega = (0,\omega) \times (0,T), \tag{35}$$

$$u(x,0) = 0, \ x \in [0,\omega], \tag{36}$$

$$u(x,T) = 0, \ x \in [0,\omega], \tag{37}$$

$$u(0,t) = \tilde{\psi}_1(t), \ u(\omega,t) = \tilde{\psi}_2(t), \ t \in [0,T], \tag{38}$$

where $\omega = 0.5, T = 0.1, a(x,t) = 1, c(x,t) = 0, b(x,t) = t^2 + 1, \varphi(x,t) = x^2, \psi(x,s) = s,\ f(x,t) = xe^{xt}\sin(10\pi t) + 10\pi e^{xt}\cos(10\pi t) - t^2 e^{xt}\sin(10\pi t) - (t^2+1)(x^3+1) - \frac{[e^{0.1}(\pi x^2 - 20\pi x + 100\pi^3) - 20\pi x]x^2}{(x^2 + 100\pi^2)^2},\ \tilde{\psi}_1(t) = \sin(10\pi t), \tilde{\psi}_2(t) = \mathrm{e}^{0.5\mathrm{t}}\sin(10\pi t)$.

We take $h = 0.1$ and produce a discretization by x: $x_i = ih, i = \overline{0,5}$.

We introduce the notations$u_i(t) = u(ih,t), \mu_i = \mu(ih), f_i(t) = f(ih,t), i = \overline{0,5}$.

Problem (35) - (38) is replaced by the following linear boundary value problem with a parameter for an integro-differential equation

$$\frac{du_i}{dt} = \frac{u_{i+1}-2u_i+u_{i-1}}{0.01} + (t^2+1)\mu_i + (0.1\cdot i)^2 \int_0^{0.1} s\cdot u_i(s)ds + f_i(t),\ \overline{1,4}, \tag{39}$$

$$u_i(0) = 0,\ i = \overline{0,5}, \tag{40}$$

$$u_i(0.1) = 0,\ i = \overline{0,5}, \tag{41}$$

$$u_0(t) = \sin(10\pi t),\ \ u_5(t) = \mathrm{e}^{0.5\mathrm{t}}\sin(10\pi t),\ \ t \in [0,0.1]. \tag{42}$$

In view of condition (40)-(42), from (39) we obtain two groups of equations for determining $\mu_0$andμ_5:

$b_0(0)\mu_0 = \dot{\tilde{\psi}}_1(0) - \varphi_0(0)\int_0^T \psi_0(s)\tilde{\psi}_1(s)ds - f_0(0)$,then$\mu_0 = 1$,

$b_5(0)\mu_5 = \dot{\tilde{\psi}}_2(0) - \varphi_5(0)\int_0^T \psi_5(s)\tilde{\psi}_2(s)ds - f_5(0)$,then$\mu_5 = 1.125$.

The functions$u_0(t),\ u_5(t)$, and parameters μ_0,μ_5 are known.

Problem (39)-(42) will be rewritten in vector-matrix form

$$\frac{du}{dt} = A(t)u + B(t)\mu + \Phi(t)\int_0^{0.1}\Psi(s)u(s)ds + F(t), \quad u,\mu \in R^4, \quad t \in (0,0.1), \tag{43}$$

$$u(0) = 0, \tag{44}$$

$$u(0.1) = 0, \tag{45}$$

where $u(t) = \big(u_1(t),u_2(t),u_3(t),u_4(t)\big)$, $\mu = (\mu_1,\mu_2,\mu_3,\mu_4)$ -unknown function and parameter,

$$A(t) = \begin{pmatrix} -200 & 100 & 0 & 0 \\ 100 & -200 & 100 & 0 \\ 0 & 100 & -200 & 100 \\ 0 & 0 & 100 & -200 \end{pmatrix},\ B(t) = \begin{pmatrix} t^2+1 & 0 & 0 & 0 \\ 0 & t^2+1 & 0 & 0 \\ 0 & 0 & t^2+1 & 0 \\ 0 & 0 & 0 & t^2+10 \end{pmatrix},$$

$$\Phi(t) = \begin{pmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.04 & 0 & 0 \\ 0 & 0 & 0.09 & 0 \\ 0 & 0 & 0 & 0.16 \end{pmatrix},\Psi(s) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix},$$

$$F(t) = \begin{pmatrix} f(0.1,t) + 100\sin(10\pi t) \\ f(0.2,t) \\ f(0.3,t) \\ f(0.4,t) + 100\mathrm{e}^{0.5\mathrm{t}}\sin(10\pi t) \end{pmatrix}.$$

Here we use thenumerical implementation of algorithm. Accuracy of solution depends on the accuracy of solving the Cauchy problem on subintervals andevaluating definite integrals. We provide the results of the numerical implementation of algorithm by partitioning theinterval [0, 0.1] with step h = 0.002.

Solution to problem (35)-(38) is pair $(u^*(x,t),\mu^*(x))$, where $u^*(x,t) = \mathrm{e}^{\mathrm{xt}}\sin(10\pi t)$, $\mu^*(x) = x^3+1$. Then solution to problem (43)-(45) is pair $(u^*(t),\mu^*)$, where $u^*(t) = \begin{pmatrix} \mathrm{e}^{0.1\mathrm{t}}\sin(10\pi t) \\ \mathrm{e}^{0.2\mathrm{t}}\sin(10\pi t) \\ \mathrm{e}^{0.3\mathrm{t}}\sin(10\pi t) \\ \mathrm{e}^{0.4\mathrm{t}}\sin(10\pi t) \end{pmatrix}$, $\mu^* = \begin{pmatrix} 1.001 \\ 1.008 \\ 1.027 \\ 1.064 \end{pmatrix}$ and the following estimates $max\|\mu^* - \tilde{\mu}\| < 0.00009$, and $\max_{j=\overline{0,50}}\|u^*(t_j) - \tilde{u}(t_j)\| < 0.000004$ is true.

УДК 517.958:536.2; 517.968.7; 519.62/.64
МРНТИ 27.35.45; 27.33.19; 27.41.19

А.Т. Асанова[1,2], Э.А. Бакирова[1,2,3], Ж.М. Кадирбаева[1,2,4]

[1]Математика және математикалық модельдеу институты, Алматы, Қазақстан;
[2]Ақпараттық және есептеуіш технологиялар институты, Алматы, Қазақстан;
[3]Қазақ Ұлттық қыздар педагогикалық университеті, Алматы, Қазақстан;
[4]Халықаралық ақпараттық технологиялар университеті, Алматы, Қазақстан

ПАРАБОЛАЛЫҚ ТЕКТЕС ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ҮШІН БАСҚАРУ ЕСЕБІН ШЕШУДІҢ САНДЫҚ ЖУЫҚТАЛҒАН ӘДІСІ

Аннотация. Параболалық тектес интегралдық-дифференциалдық теңдеулер үшін параметрі бар сызықтық шеттік есеп зерттеледі. Кеңістіктік айнымалыны дискреттеу көмегімен қарастырылатын есеп жәй интегралдық-дифференциалдық теңдеулер жүйесі үшін параметрі бар сызықтық екі нүктелі шеттік есеппен аппроксимацияланады. Алынған есепті шешу үшін параметрлеу әдісі қолданылады. Аппроксимацияланған есеп Фредгольм интегралдық дифференциалдық теңдеулер жүйесі үшін арнайы Коши есептерінен, шеттік шарттардан және бөлу нүктелерінде шешімнің үзіліссіз шарттарынан тұратын пара-пересепке келтіріледі. Параметрлері бар жәй дифференциалдық теңдеулер жүйесі үшін Коши есебін шешу дифференциалдық теңдеудің фундаменталдық матрицасы көмегімен құрылады. Параметрлерге қатысты сызықты алгебралық теңдеулер жүйесі тиісті нүктелердің мәндерін шеттік шарт пен үзіліссіз шарттарына қою арқылы құрылады.Есепті шешудің құрылған жүйе мен ішкі аралықтардаКоши есебін шешудің4-ші ретті Рунге-Куттаәдісіне негізделген сандық әдісі ұсынылады.

Кілттік сөздер: параболикалық тектес дербес туындылы интегралдық-дифференциалдық теңдеулер, параметрі бар есеп, аппроксимация, сандық жуықталған әдіс, алгоритм.

УДК 517.958:536.2; 517.968.7; 519.62/.64
МРНТИ 27.35.45; 27.33.19; 27.41.19

А.Т. Асанова[1,2], Э.А. Бакирова[1,2,3], Ж.М. Кадирбаева[1,2,4]

[1]Институт математики и математического моделирования, Алматы, Казахстан;
[2]Институт информационных и вычислительных технологий, Алматы, Казахстан;
[3]Казахский Национальный Женский Педагогический Университет, Алматы, Казахстан;
[4]Международный Университет Информационных Технологий, Алматы, Казахстан

ЧИСЛЕННО ПРИБЛИЖЕННЫЙ МЕТОД РЕШЕНИЯ ЗАДАЧИ УПРАВЛЕНИЯ ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ПАРАБОЛИЧЕСКОГО ТИПА

Аннотация. Исследуется линейная краевая задача с параметром для интегро-дифференциальных уравнений параболического типа. С помощью дискретизации пространственной переменной рассматриваемая задача аппроксимируется линейной двухточечной краевой задачей с параметром для системы интегро-дифференциальных уравнений. Для решения полученной задачи применяется метод параметризации. Аппроксимирующая задача сведена к эквивалентной задаче, состоящей из специальной задачи Коши для системы интегро-дифференциальных уравнений Фредгольма, краевых условий и условий непрерывности решения в точках разбиения.Решение задачи Коши для системы обыкновенных дифференциальных уравнений с параметрами строится с использованием фундаментальной матрицы дифференциального уравнения. Система линейных алгебраических уравнений относительно параметров составляется путем подстановки значений соответствующих точек в краевое условие и условия непрерывности. Предлагается численный метод решения задачи, основанный на решении построенной системы и метода Рунге-Кутты 4-го порядка для решения задачи Коши на подинтервалах.

Ключевые слова: интегро-дифференциальные уравнения с частными производными параболического типа, задача с параметром, аппроксимация, численно приближенный метод, алгоритм.

Information about authors:

Assanova Anar Turmaganbetkyzy - themember of Editorial Board of journal “News of the NAS RK. Physico-Mathematical Series”, Institute of Mathematics and Mathematical Modeling, Institute of Information and Computational Technologies,chief scientific researcher, Doctor of Physical and Mathematical Sciences, professor, assanova@math.kz; https://orcid.org/0000-0001-8697-8920;

Bakirova Elmira Ainabekovna - Institute of Mathematics and Mathematical Modeling, Institute of Information and Computational Technologies,leading scientific researcher, Candidate of Physical and Mathematical Sciences, ass. professor, Kazakh National Women's Teacher Training University, professor, bakirova1974@mail.ru; https://orcid.org/0000-0002-3820-5373;

Kadirbayeva Zhazira Muratbekovna - Institute of Mathematics and Mathematical Modeling, Institute of Information and Computational Technologies,leading scientific researcher, Candidate of Physical and Mathematical Sciences, International Information Technology University, assistant professor, zh.kadirbayeva@gmail.com; https://orcid.org/0000-0001-8861-4100

REFERENCES

[1] Sloan I. and Thomée V. Time discretization of an integro-differential equation of parabolic type // SIAM Journal on Numerical Analysis, 23(1986), 1052–1061.

[2] Cannon J. and LinY. Non-classical H1 projection and Galerkin methods for non-linear parabolic integro-differential equations // Calcolo, 25(1988), 187–201.

[3] Thomée V. and ZhangN.-Y. Error estimates for semidiscrete finite element methods for parabolic integro-differential equations // Mathematics of Computation, 53(1989), 121–139.

[4] LinY., ThoméeV., and WahlbinL. Ritz-Volterra projections to finite element spaces and applications to integro-differential and related equations // SIAM Journal on Numerical Analysis, 28(1991), 1047–1070.

[5] Pani A. and PetersonE. Finite element methods with numerical quadrature for parabolic integrodifferential equations // SIAM Journal on Numerical Analysis, 33(1996), 1084–1105.

[6] LinY. Semi-discrete finite element approximations for linear parabolic integrodifferential equations with integrable kernels // Journal of Integral Equations and Applications, 10(1998), 51–83.

[7] Pani A. and SinhaR. Finite element approximation with quadrature to a time dependent parabolic integro-differential equation with nonsmooth initial data // Journal of Integral Equations and Applications, 13(2001), 35–72.

[8] Pani A. and FairweatherG. H1-Galerkin mixed finite element methods for parabolic partial integro-differential equations // IMA Journal of Numerical Analysis, 22(2002), 231–252.

[9] McLean W., Sloan I. and Thomée V. Time discretization via Laplace transformation of an integro-differential equation of parabolic type // IMA Journal of Numerical Analysis, 24(2004), 439–463.

[10] Matache A.-M., Schwab C. and Wihler T. Fast numerical solution of parabolic integrodiffe-rential equations with applications in finance // SIAM Journal on Scientific Computing, 27(2005), 369–393.

[11] Sinha R., Ewing R. and LazarovR. Some new error estimates of a semidiscrete finite volume element method for a parabolic integro-differential equation with nonsmooth initial data // SIAM Journal on Numerical Analysis, 43(2006), 2320–2344.

[12] Zhang N. On fully discrete Galerkin approximations for partial integrodifferential equations of parabolic type // Mathematics of Computation, 60(1993), 133–166.

[13] Dolejší V. and Vlasák M. Analysis of a BDF-DGFE scheme for nonlinear convection-diffu-sion problems // NumerischeMathematik, 110(2008), 405–447.

[14] Volpert V. Elliptic partial differential equations Vol. 2: Reaction-Diffusing Equations, Birkhauser Springer, Basel etc., 2014.

[15] Douglas J. and Jones B. Numerical methods for integro-differential equations of parabolic and hyperbolic types // NumerischeMathematik, 4(1962), 96–102.

[16] Yanik E.and FairweatherG. Finite element methods for parabolic and hyperbolic partial integro-differential equations // Nonlinear Analysis, 12(1988), 785–809.

[17] Pani A., ThoméeV., and Wahlbin L. Numerical methods for hyperbolic and parabolic integro-differential equations// Journal of Integral Equations and Applications, 4(1992), 533–584.

[18] Dzhumabayev D.S. Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation// U.S.S.R. Computational Mathematics and Mathematical Physics. 1989. Vol. 29, No. 1. P. 34-46.

[19] Dzhumabaev D.S. A method for solving the linear boundary value problem for an integro-differential equation // Computational Mathematics and Mathematical Physics. 2010. Vol. 50, No. 7. P. 1150-1161.

[20] Dzhumabaev D.S. An algorithm for solving a linear two-point boundary value problem for an integro-differential equation // Computational Mathematics and Mathematical Physics. 2013. Vol. 53, No. 6. P. 736-758.

[21] Dzhumabaev, D.S., Bakirova, E.A. Criteria for the unique solvability of a linear two-point boundary value problem for systems of integro-differential equations // Differential Equations. 2013. Vol. 49, No. 9. P. 1087-1102.

[22] Dzhumabaev D.S. Necessary and sufficient conditions for the solvability of linear boundary-value problems for the Fredholmintegro-differential equations // Ukranian Mathematical Journal. 2015. Vol. 66, No. 8. P. 1200-1219.

[23] Dzhumabaev D.S. Solvability of a linear boundary value problem for a Fredholmintegro-differential equation with impulsive inputs // Differential Equations. 2015. Vol. 51, No. 9. P. 1180-1196.

[24] Dzhumabaev D.S. On one approach to solve the linear boundary value problems for Fredholmintegro-differential equations // Journal of Computational and Applied Mathematics. 2016. Vol. 294, P. 342-357.

[25] Dzhumabaev D.S. Bakirova, E.A., KadirbayevaZh.M. An algorithm for solving a control problem for a differential equation with a parameter // News of the NAS RK. Phys.-Math. Series. 2018. Vol. 5, No. 321. P.25-32. DOI: https://doi.org/10.32014/2018.2518-1726.4

[26] Assanova A.T., Bakirova E.A., KadirbayevaZh.M. Numerical implementation of solving a boundary value problem for a system of loaded differential equations with parameter // News of the NAS RK. Phys.-Math. Series. 2019. Vol. 3, No. 325. -P. 77-84. DOI: https://doi.org/10.32014/2019.2518-1726.27