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MULTIPERIODIC SOLUTIONS OF LINEAR SYSTEMS INTEGRO-DIFFERENTIAL EQUATIONS WITH D_c-OPERATOR AND ε-PERIOD OF HEREDITARY

Abstract. The article explores the questions of the initial problem and the problem of the multiperiodicity solutions of linear systems integro-differential equations with an operator of the form $D_c = \partial/\partial\tau + c_1\,\partial/\partial t_1 + \ldots + c_m\,\partial/\partial t_m$, $c = (c_1, \ldots, c_m) - const$ and with finite hereditary period $\varepsilon = const > 0$ that describe hereditary phenomena. Along with the equation of zeros of the operator D_c are considered linear systems of homogeneous and inhomogeneous integro-differential equations, sufficient conditions are established for the unique solvability of the initial problems for them, both necessary and sufficient conditions of multiperiodic existence are obtained by (τ, t) with periods (θ, ω) of the solutions. The integral representations of multiperiodic solutions of linear inhomogeneous systems are determined 1) in the particular case when the corresponding homogeneous systems have exponential dichotomy and 2) in the general case when the homogeneous systems do not have multiperiodic solutions, except for the trivial one.

Key words: integro-differential equation, hereditary, fluctuation, multiperiodic solution.

1. Problem statement.

In this paper, we've researched the problem of the existence of (θ, ω)-periodic solutions $u(\tau, t)$ by $(\tau, t) = (\tau, t_1, \ldots, t_m) \in R \times R \times \cdots \times R = R \times R^m$ systems of

$$D_c u(\tau, t) = A(\tau, t)u(\tau, t) + \int_{\tau-\varepsilon}^{\tau} K(\tau, t, s, t - c\tau + cs)u(s, t - c\tau + cs)ds + f(\tau, t) \qquad (1.1)$$

with a differentiation operator D_c of the form

$$D_c = \partial/\partial\tau + \langle c, \partial/\partial t\rangle, \qquad (1.2)$$

that turns into the operator of the total derivative $d/d\tau$ along the characteristics $t = c\tau - cs + \sigma$ with initial data $(s, \sigma) \in R \times R^m$, where $R = (-\infty, +\infty)$, $c = (c_1, \ldots, c_m)$ is constant vector with non-zero coordinates c_j, $j = \overline{1, m}$, $\partial/\partial t = (\partial/\partial t_1, \ldots, \partial/\partial t_m)$ is vector, $\langle c, \partial/\partial t\rangle$ is the scalar product of vectors, $A(\tau, t)$ and $K(\tau, t, s, \sigma)$ are given $n \times n$-matrices, $f(\tau, t)$ is n-vector-function, $(\theta, \omega) = (\theta, \omega_1, \ldots, \omega_m)$ is vector-period with rationally incommensurable coordinates, ε is positive constant.

The problem of this kind involves the research problems of hereditary vibrations in mechanics and electromagnetism. For example, if the oscillation phenomenon is hereditary in nature, then the equation of motion of the string at a known moment $m(\tau)$ is set by changing the angle of string torsion $\omega(\tau)$, subordinated to the ratio

$$m(\tau)-\mu\frac{d^2\omega(\tau)}{d\tau^2}=h\omega(\tau)+\int\limits_{\tau-\varepsilon}^{\tau}\varphi(\tau,s)\omega(s)ds, \tag{1.3}$$

where μ and h are constants and ε is the hereditary period of the vibrational phenomenon.

It is also known that the hereditary biological phenomenon “predator-prey” - (N_1, N_2) is related by the law of oscillations described by the system of equations

$$\begin{aligned}\frac{dN_1(\tau)}{d\tau}&=N_1(\tau)\left\{\varepsilon_1-\gamma_1N_2(\tau)-\int\limits_{\tau-\varepsilon}^{\tau}F_1(\tau-s)N_2(s)ds\right\},\\ \frac{dN_2(\tau)}{d\tau}&=N_2(\tau)\left\{-\varepsilon_2+\gamma_2N_1(\tau)+\int\limits_{\tau-\varepsilon}^{\tau}F_2(\tau-s)N_1(s)ds\right\}.\end{aligned} \tag{1.4}$$

Where $\varepsilon_1,\varepsilon_2$ and γ_1,γ_2 are constants, F_1 and F_2 are functions vanishing zero at $\tau-s\geq\varepsilon$, ε is the period of hereditary nature of the biological phenomenon under consideration.

Obviously, the above integro-differential equations (1.3) and (1.4) are particular cases of the mathematical model hereditary phenomena described by the system of equations

$$\frac{dx}{dt}=P(\tau)x(\tau)+\int\limits_{\tau-\varepsilon}^{\tau}Q(\tau,s)x(s)ds+\psi(\tau) \tag{1.5}$$

relatively sought n-vector-function with given $n\times n$-matrices $P(\tau)$ and $Q(\tau,s)$ and with n-vector-function $\psi(\tau)$, where $\varepsilon>0$ is a constant. Since the process is oscillatory, as a rule, the matrix $P(\tau)$ and the vector function $\psi(\tau)$ are almost periodic in general case and the kernel $Q(\tau,s)$ has the property of diagonal periodicity by $(\tau,s)\in R\times R$.

In particular, the indicated input data of system (1.5) are quasiperiodic by $\tau\in R$ with a frequency basis $\nu_0=\theta^{-1},\nu_1=\omega_1^{-1},\ldots,\nu_m=\omega_m^{-1}$, then in the theory of fluctuations, the question of the existence of quasiperiodic solutions $x(\tau)$ of system (1.5) with a modified frequency basis is important $\widetilde{\nu}_0=\theta^{-1},\widetilde{\nu}_1=c_1\omega_1^{-1},\ldots,\widetilde{\nu}_m=c_m\omega_m^{-1}$ and we set $\varepsilon<\theta=\omega_0<\omega_1<\ldots<\omega_m$.

An important role in solving this problem is played by the well-known theorem of G. Bohr on the deep connection between quasiperiodic functions and periodic functions of many variables (multiperiodic functions). According to this theorem, matrix-vector functions are defined $A=A(\tau,t), K=K(\tau,t,s,\sigma)$, $\sigma=t-c\tau+cs$, $f=f(\tau,t)$, $u=u(\tau,t)$ with properties of $A|_{t=c\tau}=P(\tau)$, $K|_{t=c\tau}=Q(\tau,s)$, $f|_{t=c\tau}=\psi(\tau)$, $u|_{t=c\tau}=x(\tau)$ and the operator $d/d\tau$ is replaced by a differentiation operator D_c of the form (1.2).

Thus, the problem of quasiperiodic fluctuations in systems (1.5) becomes equivalent to the problem on the existence of (θ,ω)-periodic by (τ,t) solutions $u(\tau,t)$ of the system partial integro-differential equations of the form (1.1) with differentiation operator (1.2).

The above examples of problems on string vibrations and fluctuations in the numbers of two species living together associated with the task indicate the relevance of the latter, in terms of its applicability in life. Along with this, it is worth paying special attention to the fact that the methods of researching multiperiodic solutions of integro-differential equations and systems of such partial differential equations belong to a poorly studied section of mathematics. Therefore, the development of methods of the theory of multiperiodic solutions of partial differential integro-differential equations is of special scientific interest.

In the present work are investigated to obtain conditions for the existence of multiperiodic solutions of linear systems integro-differential equations with a given differentiation operator D_c. To achieve this goal, the initial problems for the considered systems of equations are solved from the beginning, and then the necessary and sufficient conditions for the existence of multiperiodic solutions of linear systems equations are established. The integral structures of solutions linear inhomogeneous systems with the property of uniqueness are determined.

The theoretical basis of this research is based on the work of several authors. As noted above, taking into account the hereditary nature of various processes of physics, mechanics, and biology leads to the consideration of integro-differential equations [1–16], especially to the research of problems for them related to the theory of periodic fluctuations [8, 9, 12, 13]. If the heredity of the phenomenon is limited to a finite period ε of time τ, then the hereditary effect is specified by the integral operator with variable limits from $\tau - \varepsilon$ to τ.

Integro-differential equations describing phenomena with such hereditary effects are considered in [5, 6, 12, 14]. The various processes of hereditary continuum mechanics are described by partial integro-differential equations, the study of which began with the works [1, 2, 4].

The work of many authors is devoted to finding effective signs of solvability and the construction of constructive methods for researching problems for systems of differential equations, we note only [17, 18].

The research of multi-frequency oscillations led to the concept of multidimensional time. In this connection, of the theory solutions of partial differential equations that are periodic in multidimensional time is being developed, both in time and in space independent variables [19–35]. It is known that the system of canonical Hamilton equations, under fairly general conditions, can be solved by the Jacobi method, the essence of which is the transition from its integration to the integration of a partial differential equation. A similar approach is implemented in [19], where quasiperiodic solutions of ordinary differential equations are studied with a transition to the research of multiperiodic solutions of partial differential equations. This method was developed in [20–30] with its extension to the solution of a number of oscillation problems in systems of integro-differential equations.

In this research, it is examined for the first time that the problem of the existence multiperiodic solutions of systems integro-differential equations with a special differentiation operator D_c, describing hereditary processes with a finite period ε of hereditary time τ.

In solving this problem, we encountered the problems associated with the multidimensionality of time; not developed general theory of such systems; determination of structures and integral representations of solutions of linear systems equations; extending the results of the linear case to the nonlinear case; the smoothness of the solutions integral equations equivalent to the problems under consideration, etc. These barriers to solving problems have been overcome due to the spread and development of the methods of works [31-35] used to solve similar problems for systems of differential equations.

2. Zeros of the differentiation operator and their multiperiodicity.

By the zero of the operator D_c we mean a smooth function $u = u(\tau, t)$ satisfying the equation of

$$D_c u = 0. \tag{2.1}$$

The linear function

$$t = h(\tau, \tau^0, t^0) \equiv t^0 + c\tau - c\tau^0 \tag{2.2}$$

is a general solution of the characteristic equation $dt/d\tau = c$ with the initial data (τ^0, t^0), and its integral obtained from equation (2.2) by relative solution t^0 type of the form

$$h(\tau^0, \tau, t) = t - c\tau + c\tau^0 \tag{2.3}$$

is the zero of the operator D_c satisfying condition $h(\tau^0, \tau, t)\big|_{\tau=\tau^0} = t$.

It is also easy to verify that if $\psi(t)$ is an any smoothness function $e=(1,\ldots,1)$, by $t=(t_1,\ldots,t_m)\in R\times\ldots\times R=R^m$, then the function

$$u(\tau^0,\tau,t)=\psi(h(\tau^0,\tau,t)) \tag{2.4}$$

is the zero of the operator D_c satisfying condition of $u|_{\tau=\tau^0}=\psi(t)$.

Since the function $\psi(t)$ is arbitrary in the class $C_t^{(e)}(R^m)$ of functions smoothness e by $t\in R^m$, relation (2.4) is a general formula of the zeros of the operator D_c.

In connection with the research of question on multiperiodicity of the zeros operator D_c, attention should be paid to the following properties of the characteristics $h(s,\tau,t)$ of operator D_c:

$$h(s+\theta,\tau+\theta,t)=h(s,\tau,t), \tag{2.5}$$

$$h(s,\tau+\theta,t)=h(s,\tau,t)-c\theta, \tag{2.6}$$

$$h(s,\tau,t+q\omega)=h(s,\tau,t)+q\omega, \tag{2.7}$$

which follow from the linearity of the function (2.3), where $q\omega=(q_1\omega_1,\ldots,q_m\omega_m)$, $q=(q_1,\ldots,q_m)\in Z\times\ldots\times Z=Z^m$, Z are set of integers.

If $u(\tau,t)$ is the zero of operator D_c (θ,ω)-periodic by (τ,t), then the initial function $u|_{\tau=\tau^0}=u^0(t)$ is ω-periodic by t:

$$u^0(t+q\omega)=u^0(t)\in C_t^{(e)}(R^m), q\in Z^m. \tag{2.8}$$

Therefore, condition (2.8) is a necessary condition for the (θ,ω)-periodicity of zero $u(\tau,t)\in C_{\tau,t}^{(1,e)}(R\times R^m)$.

Suppose that for zero $u(\tau,t)$ of the operator D_c the necessary condition is satisfied (2.8) for its (θ,ω)-periodicity by (τ,t). Then $u(\tau,t)$ according to formula (2.4) has the form of

$$u(\tau,t)=u^0(h(\tau^0,\tau,t)). \tag{2.9}$$

Obviously, by virtue of conditions (2.7) and (2.8), the researching zero (2.9) is ω-periodic by t. For a zero $u(\tau,t)$ to be θ-periodic by τ, we require that condition

$$u^0(h(\tau^0,\tau+\theta,t))=u^0(h(\tau^0,\tau,t)-c\theta) \tag{2.10}$$

which holds by virtue of property (2.6).

From this it is clear that, under condition (2.8), relation (2.10) holds if only some integer vector $q^0\in Z^m$ is found and equality

$$c\theta+q^0\omega=0, \tag{2.11}$$

which means the commensurability of the $c\theta=(c_1\theta,\ldots,c_m\theta)$ and $\omega=(\omega_1,\ldots,\omega_m)$ vectors.

It should be noted here that condition (2.11) is required if the initial function $u^0(t)$ necessarily depends on the variable t. Otherwise, when $u^0=const$, condition (2.10) is performed automatically, conditions (2.11) are not needed.

Thus, if conditions (2.11) are not satisfied, then the (θ,ω)-periodic zero of the operator D_c are constant.

Obviously, due to the condition of (2.5), the zeros $u(\tau^0,\tau,t)$ of operator D_c form (2.4) have the property of diagonal θ-periodicity by (τ^0,τ): $u(\tau^0+\theta,\tau+\theta,t)=u(\tau^0,\tau,t)$.

The proof of this property follows from (2.5) and (2.4) based on direct verification.

The obtained results are summarized in the form of the following theorem.

Theorem 2.1. *1) If condition (2.11) is not satisfied, then only constants are the (θ,ω)-periodic zeros of the operator D_c and it does not have multiperiodic variables zeros. 2) If condition (2.11) is satisfied, then any zero of the operator D_c with an initial function of the form (2.8) is (θ,ω)-periodic, in particular, it can be any constant. 3) Any zero of the form (2.4) has the property of diagonal θ-periodicity by (τ^0,τ), and from its θ-periodicity zeros by τ follows its θ-periodicity by τ^0.*

Further, in conclusion, we note one more important group property of characteristic

$$h(\tau^0,\xi,h(\xi,\tau,t))=h(\tau^0,\tau,t), \tag{2.12}$$

necessary in the future, in justice, which can be verified by direct verification.

3. Linear homogeneous equations and their multiperiodic solutions.

We consider the initial problem for a linear homogeneous system

$$D_c u(\tau,t)=A(\tau,t)u(\tau,t)+\int\limits_{\tau-\varepsilon}^{\tau}K(\tau,t,s,h(s,\tau,t))u(s,h(s,\tau,t))ds \tag{3.1}$$

with respect to the desired n-vector-function $u(\tau,t)$ with condition

$$u(\tau,t)\big|_{\tau=\tau^0}=u^0(t)\in C_t^{(e)}(R^m) \tag{3.1^0}$$

under assumptions of

$$A(\tau+\theta,t+q\omega)=A(\tau,t)\in C_{(\tau,t)}^{(0,2e)}(R\times R^m),\ q\in Z^m, \tag{3.2}$$

$$K(\tau+\theta,t+q\omega,s,\sigma)=K(\tau,t,s+\theta,\sigma+q\omega)=K(\tau,t,s,\sigma)\in$$
$$\in C_{\tau,t,s,\sigma}^{(0,2e,0,2e)}(R\times R^m\times R\times R^m),\ q\in Z^m \tag{3.3}$$

where $\tau^0\in R$.

It is obvious [19, 20, 21, 28, 29] that under condition of (3.2), using the method of successive approximations, we can construct a matricant $W(\tau^0,\tau,t)$ of the linear system of partial differential equations of the form

$$D_c w(\tau,t)=A(\tau,t)w(\tau,t), \tag{3.4}$$

which has property

$$D_c W(\tau^0,\tau,t)=A(\tau,t)W(\tau^0,\tau,t),\ W(\tau^0,\tau^0,t)=E, \tag{3.5}$$

$$D_c W^{-1}(\tau^0,\tau,t)=-W^{-1}(\tau^0,\tau,t)A(\tau,t), \tag{3.6}$$

$$W(\tau^0+\theta,\tau+\theta,t+q\omega)=W(\tau^0,\tau,t),\ q\in Z^m, \tag{3.7}$$

where E is the identity n-matrix.

Then, using the replacement of

$$u(\tau,t)=W(\tau^0,\tau,t)v(\tau,t) \tag{3.8}$$

system (3.1) is reduced to the form of integro-differential equation

$$D_c v(\tau,t)=\int_{\tau-\varepsilon}^{\tau} Q(\tau^0,\tau,t,s,h(s,\tau,t))v(s,h(s,\tau,t))ds \tag{3.9}$$

with the kernel

$$Q(\tau^0,\tau,t,s,\sigma)=W^{-1}(\tau^0,\tau,t)K(\tau,t,s,\sigma)W(\tau^0,s,\sigma), \tag{3.10}$$

which, due to the properties (2.5)-(2.7) of the characteristics $h(s,\tau,t)$, (3.3) of the kernel $K(\tau,t,s,\sigma)$ and (3.5)-(3.7) matricant $W(\tau^0,\tau,t)$, has the properties of multiperiodicity and smoothness of the form

$$\begin{gathered} Q(\tau^0+\theta,\tau+\theta,t+q\omega,s+\theta,h(s+\theta,\tau+\theta,t+q\omega))= \\ =Q(\tau^0,\tau,t,s,h(s,\tau,t))=Q(\tau^0,\tau,t,s,\sigma)\in \\ \in C^{(1,1,e,1,e)}_{\tau^0,\tau,t,s,\sigma}(R\times R\times R^m\times R\times R^m), q\in Z^m. \end{gathered} \tag{3.11}$$

Further, under condition (3.3), integrating along the characteristics: $\tau=\eta$, $t=h(\eta,\tau,t)$, using property (2.12) of the form $h(\xi,\eta,h(\eta,\tau,t))=h(\xi,\tau,t)$, from equation (3.9), using the method of successive approximations, we find its matrix solution $V(s,\tau,t)$ based on the integral equation

$$V(s,\tau,t)=E+\int_s^{\tau} d\eta \int_{\eta-\varepsilon}^{\eta} Q(s,\eta,h(\eta,\tau,t),\xi,h(\xi,\tau,t))V(s,\xi,h(\xi,\tau,t))d\xi, \tag{3.12}$$

and by virtue of properties (3.11) of the kernel Q of this equation, we easily have the following relation

$$V(s+\theta,\tau+\theta,t+q\omega)=V(s,\tau,t)\in C^{(1,1,e)}_{s,\tau,t}(R\times R\times R^m),\ q\in Z^m. \tag{3.13}$$

Obviously, by virtue of (3.12) and (3.13), we have

$$D_c V(s,\tau,t)=\int_{\tau-\varepsilon}^{\tau} Q(s,\tau,t,\xi,h(\xi,\tau,t))V(\xi,h(\xi,\tau,t))d\xi, \tag{3.14}$$

$$V(s,s,t)=E. \tag{3.14^0}$$

We note that the matrix A, the kernel K, and the period ε are such that the matrix $V(s,\tau,t)$ is invertible, moreover

$$D_c V^{-1}(s,\tau,t)=-V^{-1}(s,\tau,t)\cdot D_c V(s,\tau,t)\cdot V^{-1}(s,\tau,t). \tag{3.14$'$}$$

Then the matrix

$$U(s,\tau,t)=W(s,\tau,t)V(s,\tau,t), \tag{3.15}$$

constructed on the basis of the replacement (3.8) is invertible, satisfies the equation (3.1), becomes the identity matrix E at $\tau=s$, and has the property of diagonal θ-periodicity by (s,τ) and ω-periodicity by t:

$$D_c U(s,\tau,t)=A(\tau,t)U(s,\tau,t)+\int_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))U(\xi,h(\xi,\tau,t))d\xi,$$

$$U(s,s,t)=E, \tag{3.16}$$

$$U(s+\theta,\tau+\theta,t+q\omega)=U(s,\tau,t)\in C^{(1,1,e)}_{s,\tau,t}(R\times R\times R^m),\ q\in Z^m. \tag{3.17}$$

These properties of (3.16) and (3.17) matrix $U(s,\tau,t)$ are consequences of the properties (3.5)-(3.7) and (3.13), (3.14), (3.14^0) the matrices $W(s,\tau,t)$ and $V(s,\tau,t)$.

The matrix $U(s,\tau,t)$ can be called the resolving operator of the integro-differential system (3.1).

Theorem 3.1. *Let conditions (3.2) and (3.3) are satisfied. Then the solution $u(\tau^0,\tau,t)$ of the initial problem (3.1)-(3.1^0) is uniquely determined by the relation*

$$u(\tau^0,\tau,t)=U(\tau^0,\tau,t)u^0\left(h(\tau^0,\tau,t)\right). \tag{3.18}$$

Proof. By condition (3.1^0), in accordance with formulas (2.8) and (2.9), the vector function $u^0\left(h(\tau^0,\tau,t)\right)$ is the zero of the operator D_c: $D_c u^0\left(h(\tau^0,\tau,t)\right)\equiv 0$.

Given this, by virtue of relations (2.12), (3.16) and expression (3.18), we have

$$D_c u^0\left(h(\tau^0,\tau,t)\right)=A(\tau,t)U(\tau^0,\tau,t)u^0\left(h(\tau^0,\tau,t)\right)+$$

$$+\int_{\tau-\varepsilon}^{\tau}K(\tau,t,\xi,h(\xi,\tau,t))U(\xi,h(\xi,\tau,t))d\xi\cdot u^0\left(h(\tau^0,\tau,t)\right)=$$

$$=A(\tau,t)u(\tau^0,\tau,t)+\int_{\tau-\varepsilon}^{\tau}K(\tau,t,\xi,h(\xi,\tau,t))U(\xi,h(\xi,\tau,t))u^0\left(h(\tau^0,\xi,h(\xi,\tau,t))\right)d\xi=$$

$$=A(\tau,t)u(\tau^0,\tau,t)+\int_{\tau-\varepsilon}^{\tau}K(\tau,t,\xi,h(\xi,\tau,t))u(\tau^0,\xi,h(\xi,\tau,t))d\xi.$$

Thus, we were convinced that the vector function (3.18) satisfies the system (3.1). By virtue of (2.3) and (3.16), at $\tau=\tau^0$ we have condition (3.1^0). The uniqueness of the solution (3.18) follows from the uniqueness of the definition of matrices $W(\tau^0,\tau,t)$ and $V(\tau^0,\tau,t)$.

The theorem is completely proved.

Now, after establishing the structure of the general solution (3.18) of system (3.1), we have the opportunity to research the multiperiodicity of its solutions.

Theorem 3.2. *Let the conditions of theorem 3.1 are satisfied. In order to the solution $u(\tau,t)$ of system (3.1) is (θ,ω)-periodic, it is necessary, that its initial function $u(0,t)=u^0(t)$ at $\tau=0$ should be ω-periodic continuously differentiable function of the variable $t\in R^m$:*

$$u^0(t+q\omega)=u^0(t)\in C_t^{e/\omega}(R^m),\ q\in Z^m. \tag{3.19}$$

Proof. Indeed, for $\tau^0=0$, from the formula for the general solution (3.18) of system (3.1) we have

$$u(\tau,t)=U(0,\tau,t)u^0\left(h(0,\tau,t)\right), \tag{3.20}$$

and it is (θ,ω)-periodic by (τ,t). Therefore, the condition is satisfied

$$u(\tau+\theta,t+q\omega)=u(\tau,t),\ q\in Z^m. \tag{3.21}$$

Then, in particular, from the set (3.21) we obtain relation of

$$u(\tau,t+q\omega)=u(\tau,t),\ q\in Z^m. \tag{3.22}$$

Substituting the representation (3.20) into the identity (3.22) we have

$$U(0,\tau,t+q\omega)u^0\left(h(0,\tau,t+q\omega)\right)=U(0,\tau,t)u^0\left(h(0,\tau,t)\right).$$

Then, by virtue of properties (2.7) and (3.17), we obtain

$$U(0,\tau,t)u^0(h(0,\tau,t)+q\omega)=U(0,\tau,t)u^0(h(0,\tau,t)),\ q\in Z^m.$$

Further, setting the $t=0$ and taking into account (3.16), we have

$$u^0(t+q\omega)=u^0(t),\ q\in Z^m.$$

Thus, the identity (3.19) is proved. The smoothness of the initial functions $u^0(t)$ follows from the smoothness of the solution $u(\tau,t)$ of system (3.1) itself. This is what was required to prove.

Theorem 3.3. *In order for the solution $u(\tau,t)$ of system (3.1) for being ω-periodic by $t\in R^m$ under the conditions of theorem 3.2, it is necessary and sufficient for condition (3.19) be satisfied with respect to the initial function $u^0(t)$ for $\tau=0$.*

Proof. ◄ Necessity follows from theorem 3.2. To prove sufficiency, we show that relation (3.22) follows from condition (3.19). To do this, it suffices to use representation (3.20) and properties (2.7) and (3.17) of the characteristic and matricant, respectively. ►

Theorem 3.4. *In order for the solution $u(\tau,t)$ to be θ-periodic by $\tau\in R$ under the conditions of theorem 3.3, it is necessary and sufficient that the initial function $u^0(t)$ at $\tau=0$ be a ω-periodic solution of the linear ω-periodic by t functional difference system*

$$U(0,\theta,t)u^0(t-c\theta)=u^0(t) \tag{3.23}$$

with difference $\rho=c\theta$ by t.

◄ The condition of θ-periodicity by τ of the solution $u(\tau,t)$ has the form

$$u(\tau+\theta,t)=u(\tau,t),\ (\tau,t)\in R\times R^m\ . \tag{3.24}$$

By virtue of the uniqueness properties, the solutions of system (3.1) to satisfy condition (3.24) are necessary and sufficient for condition

$$u(\theta,t)=u(0,t) \tag{3.25}$$

to hold.

Then, using representation (3.20), we rewrite identity (3.25) in the form

$$U(0,\theta,t)u^0(h(0,\theta,t))=U(0,0,t)u^0(h(0,0,t)).$$

Hence, by virtue of properties (2.6) and (3.16), we have the necessary and sufficient condition (3.23).►

If $u_0(\tau,t)=u^0(h(0,\tau,t))$ is the (θ,ω)-periodic zero of the operator D_c, then the solution $u(\tau,t)$ of the form (3.20):

$$u(\tau,t)=U(0,\tau,t)u_0(\tau,t) \tag{3.26}$$

we call the solution ω-periodic by t generated by the (θ,ω)-periodic zero $u_0(\tau,t)$ of the differentiation operator D_c.

Theorem 3.5. *In order for the solution $u(\tau,t)$ to be (θ,ω)-periodic solution of system (3.1) generated by the (θ,ω)-periodic zero $u_0(\tau,t)$ of the operator D_c under the conditions of theorem 3.4, it is necessary and sufficient that the vector function $u_0(\tau,t)=v(t)$ be an eigenvector of the monodromy matrix $U(0,\theta,t)=V(t)$:*

$$[V(t)-E]v(t)=0. \tag{3.27}$$

◄ According to theorem 2.1, we have $u^0(t-c\theta)=u^0(t)$. Therefore, the necessary and sufficient condition (3.23) given by theorem 3.4 has the form

$$[U(0,\theta,t)-E]u^0(t)=0. \tag{3.28}$$

Obviously, $u^0(t)=u_0(0,t)=v(t)$. Then from relation (3.28) we have the condition of (θ,ω)-periodicity of solutions $u(\tau,t)$ from the class under consideration generated by the multiperiodic zeros of operator D_c.►

Note that if the commensurability condition (2.11) is not satisfied, then $v(t)$ becomes constant: $v=c^0-const$ and the condition (3.27) of multiperiodicity has the form

$$[V(t)-E]c^0=0,\ \ t\in R^m. \tag{3.29}$$

In order to avoid nonzero (θ,ω)-periodic solutions of system (3.1), in this case, it is sufficient to require that condition

$$\det[V(t)-E]\neq 0,\ \ t\in R^m. \tag{3.30}$$

be satisfied.

The research of multiperiodic solutions of the form (3.26) of system (3.1) is a separate interesting direction in the theory of multiperiodic solutions of such systems, which is based on conditions (3.27) - (3.30).

In many cases, there is necessary to clarify the conditions for the absence of multiperiodic solutions of systems of the form (3.1) other than the trivial $u=0$.

To do this, according to theorem 3.4, it is necessary to require that the ω-periodic functional-difference system (3.23) does not have a solution nonzero that is ω-periodic by t. In this regard, we assume that the resolving operator $U(\tau^0,\tau,t)$ of the system of integro-differential equations (3.1) satisfies condition

$$|U(s,\tau,t)|\leq a\,e^{-\alpha(\tau-s)},\ \tau\geq s \tag{3.31}$$

with the constants $a\geq 1$ and $\alpha>0$.

Then the matrix $U(0,\tau,t)$ at $\tau=0$ turns into the identity matrix E and at $\tau>0$, according to condition (3.31), decreases exponentially. Therefore, the monodromy matrix $U(0,\theta,t)=V(t)$ at all $t\in R^m$ satisfies condition

$$|V(t)|\leq b=const<1,\ \ t\in R^m, \tag{3.32}$$

where $b=a\,e^{-\alpha\theta}=const<1$.

Therefore, representing system (3.23) in the form

$$u^0(t)=V(t)u^0(t-c\theta) \tag{3.33}$$

and integrating it j times, we have

$$u^0(t)=V(t)V(t-c\theta)\ldots V(t-jc\theta)u^0(t-c\theta-jc\theta).$$

Estimating the last relation, on the basis of (3.32) we have inequality

$$|u^0(t)|\leq b^{j+1}|u^0(t-c\theta-jc\theta)|,\ \ 0<b<1.$$

Hence, passing to the limit at $j \to +\infty$, taking into account that the quantity $\left|u^0(t)\right|$ is bounded due to its ω-periodicity, we have $\left|u^0(t)\right| = 0$, that is, system (3.33) has only a zero ω-periodic solution.

Thus, the following theorem is proved.

Theorem 3.6. *In order for the system of integro-differential equations (3.1) has no multiperiodic solutions, except for the zero one under the conditions of theorem 3.4, the fulfillment of condition (3.31) is sufficient.*

Note that the proved theorem 3.6 remains valid if condition (3.31) is replaced by condition

$$\left|U(s,\tau,t)\right| \le a\, e^{\alpha(\tau-s)}, \ \ \tau \le s \tag{3.34}$$

with constants $a \ge 1$ and $\alpha > 0$.

The more general than (3.31) and (3.34) the absence of condition a multiperiodic solution other than zero is the decomposability condition for a resolving operator $U(s,\tau,t)$ into the sum of two matrix solutions $U_-(s,\tau,t)$ and $U_+(s,\tau,t)$ of system (3.1) in the form

$$U(s,\tau,t) = U_-(s,\tau,t) + U_+(s,\tau,t), \tag{3.35}$$

$$D_c U_\mp(s,\tau,t) = A(\tau,t)U_\mp(s,\tau,t) + \int\limits_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))U_\mp(s,\xi,h(\xi,\tau,t))d\xi\,, \tag{3.36}$$

satisfying conditions

$$\left|U_-(s,\tau,t)\right| \le a\, e^{-\alpha(\tau-s)}, \ \tau \ge s\,, \tag{3.37}$$

$$\left|U_+(s,\tau,t)\right| \le a\, e^{\alpha(\tau-s)}, \ \tau \le s \tag{3.38}$$

with some constants $a \ge 1$ and $\alpha > 0$.

In particular, when one of the matrices U_- and U_+ is equal to zero, then we obtain either condition (3.31) or condition (3.34), respectively.

If conditions (3.35) - (3.38) are satisfied, they say that the resolving operator $U(s,\tau,t)$ has the property of exponential dichotomy.

We note that for system (3.1), the case of exponential dichotomy is possible when for the monodromy matrix U there exist projectors P_- and P_+ with the properties $P_- + P_+ = I$ is the identity operator, $P_+P_- = P_-P_+ = 0$ is the zero operator and $P_\mp(Uu) = P_\mp U \cdot P_\mp u$, where P_- projects the space of solutions onto the subspace of exponentially decreasing in norm of solutions, and P_+ - on the subspace of exponentially increasing solutions.

Then system

$$V(t)v(t-c\theta) = v(t) \tag{3.39}$$

is equivalent to system

$$V_-(t)v_-(t-c\theta) = v_-(t), \tag{3.40_-}$$

$$V(t)_+ v_+(t-c\theta) = v_+(t). \tag{3.40_+}$$

As above, it was justified that systems (3.40_-) and (3.40_+) have only zero multiperiodic solutions; therefore, system (3.39) also has only a zero solution, provided that conditions (3.35) - (3.38) are satisfied.

Thus, we can state a theorem that generalizes theorem 3.6 proved above.

Theorem 3.7. *Let conditions (3.2), (3.3), and (3.35) - (3.38) be satisfied. Then system (3.1) has no multiperiodic solutions, except for the trivial one.*

4. Linear inhomogeneous equations and their multiperiodic solutions.

We consider the linear inhomogeneous system of integro-differential equations

$$D_c u(\tau,t) = A(\tau,t)u(\tau,t) + \int_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))u(\xi,h(\xi,\tau,t))d\xi + f(\tau,t), \quad (4.1)$$

the corresponding system (3.1), where the $f(\tau,t)$ is given n-vector-function possessing property

$$f(\tau+\theta, t+q\omega) = f(\tau,t) \in C_{\tau,t}^{(0,e)}(R \times R^m), \ q \in Z^m. \quad (4.2)$$

Under the condition (4.2), we are posing the definition of a solution $u = u(\tau,t)$ of system (4.1) satisfying the initial condition

$$u\big|_{\tau=\tau^0} = u^0(t) \in C_t^{(e)}(R^m). \quad (4.1^0)$$

We begin the solution of this question about finding the particular solution $u^*(\tau^0,\tau,t)$ of system (4.1) with zero initial condition

$$u^*(\tau^0,\tau,t)\big|_{\tau=\tau_0} = 0. \quad (4.1^*)$$

We will seek this solution in the form

$$u^*(\tau^0,\tau,t) = \int_{\tau^0}^{\tau} U(s,\tau,t)v(s,h(s,\tau,t))ds \quad (4.3)$$

with an unknown, continuous, smooth by t at $(\tau,t) \in R \times R^m$ n-vector-function

$$v(\tau,t) \in C_{\tau,t}^{(0,e)}(R \times R^m), \quad (4.4)$$

where $U(s,\tau,t)$ is resolving operator of the homogeneous system (3.1).

Acting by the operator D_c on the vector function (4.3), taking into account (4.4), we have

$$D_c u^*(\tau^0,\tau,t) = \int_{\tau^0}^{\tau} D_c U(s,\tau,t)\cdot v(s,h(s,\tau,t))ds + v(\tau,t) =$$

$$= A(\tau,t)\int_{\tau^0}^{\tau} U(s,\tau,t)v(s,h(s,\tau,t))ds +$$

$$+ \int_{\tau^0}^{\tau}\left(\int_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))U(s,\xi,h(\xi,\tau,t))d\xi\right)v(s,h(s,\tau,t))ds + v(\tau,t) =$$

$$= A(\tau,t)u^*(\tau^0,\tau,t) +$$

$$+ \int_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))\left\{\int_{\tau^0}^{\tau} U(s,\xi,h(\xi,\tau,t))v(s,h(s,\xi,h(\xi,\tau,t)))ds\right\}d\xi + v(\tau,t) =$$

$$= A(\tau,t)u^*(\tau^0,\tau,t) + \int_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))u^*(\tau^0,\xi,h(\xi,\tau,t))d\xi + v(\tau,t). \quad (4.5)$$

Substituting expressions (4.3) and (4.5) into system (4.1) we obtain that

$$v(\tau,t) = f(\tau,t). \quad (4.6)$$

Then, by virtue of (4.6), we find the desired solution

$$u^*\left(\tau^0,\tau,t\right)=\int_{\tau^0}^{\tau}U(s,\tau,t)f(s,h(s,\tau,t))ds\,. \tag{4.7}$$

Obviously, the solution (4.7) satisfies condition (4.1^*).

Since the solution $u\left(\tau^0,\tau,t\right)$ of the linear inhomogeneous equation consists of the sum of the solutions of the homogeneous equation and some particular solution of the inhomogeneous equation, we have the expression for the general Cauchy solution

$$u\left(\tau^0,\tau,t\right)=U\left(\tau^0,\tau,t\right)u^0\left(h\left(\tau^0,\tau,t\right)\right)+u^*\left(\tau^0,\tau,t\right) \tag{4.8}$$

system (4.1) with initial condition (4.1^0).

Thus, we have the theorem on solving the initial problem for a linear inhomogeneous system of integro-differential equations (4.1).

Theorem 4.1. *Under conditions (3.2), (3.3) and (4.2), the initial problem (4.1) - (4.1^0) has the unique solution in the form (4.7) - (4.8).*

◄ The existence of a solution $u\left(\tau^0,\tau,t\right)$ under the conditions of the theorem is proved by the deductions of formulas (4.7) and (4.8). The uniqueness of the solution (4.8) follows from the uniqueness of solution of the initial problem for the homogeneous system (3.1).►

Now we will research the problems of multiperiodic solutions of system (4.1). Suppose that the conditions of theorem 3.7 are satisfied.

Then the homogeneous system (3.1) corresponding to system (4.1) does not have (θ,ω)-periodic non-zero solutions, and it has the property of exponential dichotomy.

In this case, the question of the existence of multiperiodic solutions of system (4.1) is investigated on the basis of the Green's function method.

To construct the Green's function $G(s,\tau,t)$, using the property of exponential dichotomy, we set

$$G(s,\tau,t)=\begin{cases}U_-(s,\tau,t), & \tau\ge s,\\ -U_+(s,\tau,t), & \tau<s,\end{cases} \tag{4.9}$$

where $U_-(s,\tau,t)$ and $U_+(s,\tau,t)$ are the terms of the resolving operator $U(s,\tau,t)$, which consists of their sum (3.35).

The constructed Green function (4.9) possess the following properties.

$$1^0.\,D_cG(s,\tau,t)=A(\tau,t)G(s,\tau,t)+$$
$$+\int_{\tau-\varepsilon}^{\tau}K(\tau,t,\xi,h(\xi,\tau,t))G(s,\xi,h(\xi,\tau,t))d\xi,\tau\ne s\,. \tag{4.10}$$

This property follows from property (3.36) of the operator $U(s,\tau,t)$.

$$2^0.\;G(s-0,\tau,t)-G(s+0,\tau,t)=E\,. \tag{4.11}$$

Lets note that follows from the equality $U_-(\tau-0,\tau,t)+U(\tau+0,\tau,t)=$
$=U_-(\tau,\tau,t)+U_+(\tau,\tau,t)=U(\tau,\tau,t)=E$.

$$3^0.\;G(s+\theta,\tau+\theta,t+q\omega)=G(s,\tau,t),\;q\in Z^m\,. \tag{4.12}$$

This property is the consequence of property (3.17) of the resolving operator $U(s,\tau,t)$.

$$4^0.\left|G(s,\tau,t)\right|\le a\,e^{-\alpha|\tau-s|},\;a\ge1\text{ and }\alpha>0\,. \tag{4.13}$$

We have this estimate from inequalities (3.37) - (3.38).

Theorem 4.2. *Suppose that the conditions of theorem 4.1 are satisfied and the matrix* $A(\tau,t)$ *with kernel* $K(\tau,t,s,\sigma)$ *are such that the linear homogeneous system (3.1) has the property of exponential dichotomy, expressed by the relation (3.35) - (3.38). Then system (4.1) has the unique* (θ,ω)*-periodic solution*

$$u^*(\tau,t)=\int_{-\infty}^{+\infty}G(s,\tau,t)f(s,h(s,\tau,t))ds, \tag{4.14}$$

satisfying estimate

$$\|u^*\|\le\frac{a}{\alpha}\|f\|, \tag{4.15}$$

where $\|u\|=\sup\limits_{R\times R^m}|u(\tau,t)|$.

◄ The convergence of the integral (4.14) and the differentiability of the solution (4.14) follow from the differentiability of the matrix-vector functions G, f and estimate (4.13). By virtue of (4.10) and (4.11), it is proved that the vector-function (4.14) satisfies system (4.1). Multiperiodicity follows from properties (4.12) and (4.2). Inequality (4.15) is the consequence of the estimate (4.13). The exponential dichotomy of system (3.1) ensures the uniqueness of the (θ,ω)-periodic solution of system (4.1). ►

Lemma 4.1. *Let the homogeneous linear system (3.1) under conditions (3.2), (3.3) and (4.2) have no* (θ,ω)*-periodic solutions except zero. Then the corresponding inhomogeneous linear system (4.1) can have at most one* (θ,ω)*-periodic solution.*

◄ Suppose that under the conditions of this lemma, system (4.1) has two different (θ,ω)-periodic solutions $u_1(\tau,t)$ and $u_2(\tau,t)\ne u_1(\tau,t)$. Then their difference $u(\tau,t)=u_2(\tau,t)-u_1(\tau,t)\ne 0$ is a (θ,ω)-periodic solution of the linear homogeneous system (3.1), which has only a zero (θ,ω)-periodic solution. The obtained contradiction proves the validity of the lemma. ►

Assuming that the ω-periodic initial function $u^0(t)\in C_t^{(e)}(R^m)$ of the (θ,ω)-periodic solution $u(\tau,t)$:

$$u(\tau,t)=U(0,\tau,t)u^0(h(0,\tau,t))+u^*(0,\tau,t), \tag{4.16}$$

represented by (4.8) has property

$$u^0(t-c\theta)=u^0(t), \tag{4.17}$$

it can also be represented by formula

$$u(\tau,t)\equiv u(\tau+\theta,t)=U(0,\tau+\theta,t)u^0(h(0,\tau,t))+u^*(0,\tau+\theta,t), \tag{4.18}$$

since, by condition (4.17) $u^0(h(0,\tau,t))$ is (θ,ω)-periodic zero operator D_c.

Then, eliminating the unknown initial function $u^0(t)$ from relations (4.16) and (4.18), we obtain

$$u(\tau,t)=\left[U^{-1}(0,\tau+\theta,t)-U^{-1}(0,\tau,t)\right]\times$$
$$\times\left\{U^{-1}(0,\tau+\theta,t)u^*(0,\tau+\theta,t)-U^{-1}(0,\tau,t)u^*(0,\tau,t)\right\}. \tag{4.19}$$

Further, using representation (4.7) of the solution $u^*(0,\tau,t)$, accepting the notation $\widetilde{U}(s,\tau,t)=U^{-1}(0,\tau,t)U(s,\tau,t)$ and setting

$$\widetilde{U}_\theta(s,\tau,t)=\begin{cases}\widetilde{U}(s,\tau,t), & \tau \xrightarrow{s} 0,\\ \widetilde{U}(s,\tau+\theta,t), & 0 \xrightarrow{s} \tau+\theta,\end{cases} \tag{4.20}$$

$$f_\theta(s,\tau,h(s,\tau,t))=\begin{cases}f(s,\tau,h(s,\tau,t)), & \tau \xrightarrow{s} 0,\\ f(s,\tau+\theta,h(s,\tau+\theta,t)), & 0 \xrightarrow{s} \tau+\theta,\end{cases} \tag{4.21}$$

we can represent relation (4.19) in the form

$$u(\tau,t)=\left[U^{-1}(0,\tau+\theta,t)-U^{-1}(0,\tau,t)\right]^{-1}\int_\tau^{\tau+\theta}\widetilde{U}_\theta(s,\tau,t)f_\theta(s,h(s,\tau,t))ds\,. \tag{4.22}$$

Thus, looking for a (θ,ω)-periodic solution of system (4.1) among solutions $u(\tau,t)$ with initial conditions having property (4.17), we showed that it is determined by formula (4.22), which is revealed by relations (4.7) and (4.19) - (4.21).

Theorem 4.3. *Suppose that conditions (3.2), (3.3), (4.2) are satisfied and the linear homogeneous system (3.1) has no (θ,ω)-periodic solutions, except for the trivial one. Then the system of inhomogeneous linear integro-differential equations (4.1) has the unique (θ,ω)-periodic solution $u(\tau,t)$ of the form (4.22).*

◄ The conclusion of the solution (4.22) is given above. Therefore, the existence of the (θ,ω)-periodic solution of system (4.1) is proved. Uniqueness follows from Lemma 4.1. ►

Note that the above researched problems for the considered systems can be considered along the characteristics $t=t^0+c\tau-c\tau^0$ with fixed initial data (τ^0,t^0).

Then, due to the fact that the operator D_c turns into the operator $d/d\tau$ of the full derivative, from the theorems proved as a corollary we have the corresponding statements about the existence of solutions to the initial problems for systems of ordinary integro-differential equations and the theorem about the existence of their Bohr quasiperiodic solutions generated by multiperiodic solutions of the original systems which we will not dwell on here.

Conclusion.

First of all, we've noted that this article proposes the method for (research) researching solutions of problems that satisfy the initial conditions and have the property of multiperiodicity with given periods for linear systems of integro-differential equations with D_c partial differential operator, ε-hereditary effect and the linear integral operator. This technique is a generalization of methods and solutions of similar problems for systems of partial differential equations with the operator D_c. The solution of problems under consideration for such systems in this formulation are researched for the first time. The relevance of the main problem is substantiated. The solutions of all the subtasks analyzed to achieve the goal are formulated as theorems with proofs. Scientific novelties include the multi-periodicity theorems of zeros of the operator D_c; about solutions to initial problems for all considered types of systems; about necessary as well as sufficient conditions for the existence of multiperiodic solutions of both homogeneous and inhomogeneous systems, as well as the integral representations of solutions inhomogeneous systems in two cases when the corresponding homogeneous systems have the properties: 1) exponential dichotomy and 2) the absence of non-trivial multiperiodic solutions, at all.

We've also noted that the consequences deduced by examining the results obtained along the characteristics $t=t^0+c\tau-c\tau^0$ with fixed (τ^0,t^0) refer to their applications in the theory of quasiperiodic solutions of systems ordinary integro-differential equations.

The technique that developed here is quite applicable to the research of problems of hereditary-string vibrations and the "predator-prey" given in the delivered part of the work, which can be attributed to examples of the applied aspect.

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D_c-ОПЕРАТОРЛЫ ЖӘНЕ ε-ЭРЕДИТАРЛЫҚ ПЕРИОДТЫ СЫЗЫҚТЫ ИНТЕГРАЛДЫ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІНІҢ КӨППЕРИОДТЫ ШЕШІМДЕРІ

Аннотация. Мақалада $D_c = \partial/\partial\tau + c_1\,\partial/\partial t_1 + \ldots + c_m\,\partial/\partial t_m$ операторлы, $c = (c_1, \ldots, c_m) - const$ және тұқым қуалаушылық сипаттағы құбылыстарды сипаттайтын ε ақырлы эредитарлық периодты сызықты интегралды-дифференциалдық теңдеулер жүйесінің көппериодты шешімдері жөніндегі есептер мен бастапқы есеп мәселелері зерттеледі. D_c операторының нөлдерінің теңдеуімен қатар сызықты біртекті және біртекті емес интегралды-дифференциалдық теңдеулер жүйесі қарастырылды, олар үшін бастапқы есептердің бірмәнді шешілімділігінің жеткілікті шарттары анықталған, (τ, t) бойынша (θ, ω) периодты, көппериодты шешімдердің бар болуының қажетті де, жеткілікті де шарттары алынған. Сызықты біртекті емес жүйенің көппериодты шешімдерінің интегралдық өрнектері 1) дербес жағдайда, яки теңдеуге сәйкес біртекті жүйелер экспоненциалды дихотомиялық қасиетке ие болғанда және 2) жалпы жағдайда, біртекті жүйелердің нөлден басқа көппериодты шешімдері болмағанда айқындалды.

Түйін сөздер: интегралды-дифференциалдық теңдеу, эредитарлық, флуктуация, көппериодты шешім.

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МНОГОПЕРИОДИЧЕСКИЕ РЕШЕНИЯ ЛИНЕЙНЫХ СИСТЕМ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С D_c-ОПЕРАТОРОМ И ε-ПЕРИОДОМ ЭРЕДИТАРНОСТИ

Аннотация. В заметке исследуются вопросы начальной задачи и задачи о многопериодичности решений линейных систем интегро-дифференциальных уравнений с оператором вида $D_c = \partial/\partial\tau + c_1\,\partial/\partial t_1 + \ldots + c_m\,\partial/\partial t_m$, $c = (c_1, \ldots, c_m) - const$ и конечным периодом эредитарности $\varepsilon = const > 0$, которые описывают явления наследственного характера. Наряду с уравнением нулей оператора D_c рассмотрены линейные системы однородных и неоднородных интегро-дифференциальных уравнений, для них установлены достаточные условия однозначной разрешимости начальных задач, получены как необходимые, так и достаточные условия существования многопериодических по (τ, t) с периодами (θ, ω) решений. Определены интегральные представления многопериодических решений линейных неоднородных систем 1) в частном случае, когда соответствующие однородные системы обладают экспоненциальной дихотомичностью и 2) в общем случае, когда однородные системы не имеют многопериодических решений, кроме тривиального.

Ключевые слова: интегро-дифференциальное уравнение, эредитарность, флуктуация, многопериодическое решение.

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