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EQUIVALENCE OF THE HUNTER-SAXON EQUATION AND THE GENERALIZED HEISENBERG FERROMAGNET EQUATION

Abstract. Integrable systems play an important role in modern mathematics, theoretical and mathematical physics. The display of integrable equations with exact solutions and some special solutions can provide important guarantees for the analysis of its various properties. The Hunter-Saxton equation belongs to the family of integrable systems. The extensive and interesting mathematical theory, underlying the Hunter-Saxton equation, creates active mathematical and physical research. The Hunter-Saxton equation (HSE) is a high-frequency limit of the famous Camassa-Holm equation. The physical interpretation of HSE is the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field.

In this paper, we propose a matrix form of the Lax representation for HSE in $su(n+1)/s(u(1)\oplus u(n))$ - symmetric space for the case $n=2$. Lax pairs, introduced in 1968 by Peter Lax, are a tool for finding conserved quantities of integrable evolutionary differential equations. The Lax representation expands the possibilities of the equation we are considering. For example, in this paper, we will use the matrix Lax representation for the HSE to establish the gauge equivalence of this equation with the generalized Heisenberg ferromagnet equation (GHFE). The famous Heisenberg Ferromagnet Equation (HFE) is one of the classical equations integrable through the inverse scattering transform. In this paper, we will consider its generalization. Andalso the connection between the decisions of the HSE and the GHFE will be presented.

Key words: integrable systems, the Hunter-Saxton equation, generalized Heisenberg ferromagnet equation, the Lax representation, gauge equivalence.

Introduction. The equation describing an asymptotic model of liquid crystals was introduced by J.K. Hunter and R. Saxton [1] in the form

$$(u_t+uu_x)_x=\frac{1}{2}u_x^2, \tag{1}$$

$$m_t+um_x+2u_xm=0, \tag{2}$$

$$m=-u_{xx}, \tag{3}$$

where $u=u(x,t)$ and $m=m(x,t)$ - real functions.

J.K. Hunter and Y. Zheng Hunter and Zheng introduced the Hamiltonian structure of the HSE, and they also proved the complete integrability of this equation [2]. Further, in work [3], was presented solutions to the inverse scattering problem for HSE. In later works [4-8], analytical and geometric interpretations were reduced to the equation (1).

Lax Representation of the Hunter-Saxton Equation

As mentioned above, the HSE (2), (3) is integrable using the inverse scattering method and has a pseudo-differential formulation of a Lax representation [9] in the form

$$\Phi_{xx}=\lambda m\Phi, \tag{4}$$

$$\Phi_t = \left(\frac{1}{2\lambda} - u\right)\Phi_x + \frac{1}{2}u_x\Phi, \tag{5}$$

where $\Phi(x,t) = \Phi(\Phi_1, \Phi_2)^T$ - eigenfunction and, accordingly, λ -eigenvalues.

To establish the gauge equivalence between HSE and GFHE (spin system), we need the Lax matrix representation. Since the coefficients of linear systems corresponding to the spin equations are related to the symmetric matrix Lax representations [10].

Definition 1. (Lax Equations). Let $\Phi(x,t,\lambda)$ be $SU(2)$ valued function such that $(x,t) \in O \subset \Re^2$ and $\lambda \in C$ is spectral parameter. Lax equations are defined as

$$\Phi_x = U\Phi, \ \Phi_t = V\Phi, \tag{6}$$

where $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are $su(2)$ valued functions and they satisfy the following equation

$$U_t - V_x + [U,V] = 0. \tag{7}$$

Equation (7) is the compatibility condition of the (6). The matrices U and V are known as Lax pairs.

Proposition 1.1 The Lax representation for the HSE (2), (3) in symmetric space $su(n+1)/s(u(1)\oplus u(n))$ at $n = 2$ is given in the form

$$\Phi_x = U_2\Phi, \tag{8}$$

$$\Phi_t = V_2\Phi, \tag{9}$$

where

$$U_2 = \begin{pmatrix} 0 & 1 \\ \lambda m & 0 \end{pmatrix}, \ V_2 = \begin{pmatrix} \frac{u_x}{2} & \frac{1}{2\lambda} - u \\ \frac{u_{xx}}{2} + (\frac{1}{2\lambda} - u)\lambda m & -\frac{u_x}{2} \end{pmatrix}. \tag{10}$$

Proof. From the compatibility condition for system (4), (5) matrices $U_2(x,t,\lambda)$ and $V_2(x,t,\lambda)$ satisfy the zero curvature condition

$$U_{2t} - V_{2x} + [U_2, V_2] = 0. \tag{11}$$

Let us calculate the necessary components of the equation (11)

$$U_t = \begin{pmatrix} 0 & 0 \\ \lambda m_t & 0 \end{pmatrix}, \ V_x = \begin{pmatrix} \frac{u_{xx}}{2} & -u_x \\ \frac{1}{2}(u_{xxx} + m_x) - \lambda(u_x m + u m_x) & -\frac{u_{xx}}{2} \end{pmatrix},$$

$$[U,V] = \begin{pmatrix} \frac{u_{xx}}{2} & -u_x \\ \lambda m u_x & -\frac{u_{xx}}{2} \end{pmatrix},$$

and substitute it into the zero curvature condition. Equating the corresponding elements of the second rows and first columns of the matrices in the equations (11), we obtain

$$\lambda m_t - \frac{u_{xxx}}{2} - \left[\left(\frac{1}{2\lambda} - u\right)\lambda m\right]_x + \lambda m u_x = 0. \tag{12}$$

The rest of the elements will be identically equal to zero. Collecting the terms of the equation (12) by degree λ, we get the HSE (2), (3)

$$\lambda^0: \quad -\frac{1}{2}(u_{xxx} + m_x) = 0,$$

$$m_x = -u_{xxx},$$

$$m = -u_{xx},$$
$$\lambda^1: \quad m_t + u_x m + u m_x + m u_x = 0,$$
$$m_t + 2mu_x + um_x = 0.$$

Thereby the proposition 1 is proved.

A Generalized Heisenberg Ferromagnet Equation

In this subsection, we present one of the integrable generalized Heisenberg ferromagnet equations, which has the form

$$[A, A_{xt} + (uA_x)_x] = 0, \tag{13}$$

where $A = (A_1, A_2, A_3)$ is the spin vector and $A^2 = I$.

Here the real function u is expressed in terms of the spin matrix A as follows:

$$u = \frac{1}{8\beta^2}\partial_x^{-2}\left(tr(A_x^2)\right), \tag{14}$$

where $\beta = const$ and

$$A = \begin{pmatrix} A_3 & A^- \\ A^+ & -A_3 \end{pmatrix}, \ A^{\pm} = A_1 \pm iA_2. \tag{15}$$

The Lax representation corresponding to the GHFE looks as follows

$$\Psi_x = U_1\Psi, \tag{16}$$
$$\Psi_t = V_1\Psi, \tag{17}$$

here

$$U_1 = \frac{(\beta-\lambda)}{2}\left[A, \left(u - \frac{1}{2\beta}\right)A_x + A_t\right], \tag{18}$$

$$V_1 = \frac{(\beta-\lambda)}{2}\left(\frac{1}{2\lambda} - u\right)[A, A_t] + \frac{(\beta-\lambda)u}{2}\left(\frac{1}{2\lambda} + \frac{1}{2\beta} - u\right)[A, A_x]. \tag{19}$$

Gauge equivalence HSE and GHFE

Definition 2. Two systems of nonlinear equations, integrable using the inverse scattering method, are called gauge-equivalent if the corresponding flat connections U_j, V_j, $j = 1,2$ are definedin one bundle and are obtained from each other by a gauge transformation independent of λ, that is, if

$$U_1 = gU_2g^{-1} + g_xg^{-1}, V_1 = gV_2g^{-1} + g_tg^{-1},$$

where $g(x,t) \in GL(n,C)$. It is clear that in this case, in the corresponding systems of linear differential equation $\Phi_1 = g\Phi_2$.

Proposition 2.2 The HSE (2), (3) with matrix Lax representation (8), (9) and the GHFE (13) with Lax representation (16), (17)are gauge equivalent to each other[10, 11].

Proof.Based on the classical theory of gauge equivalence [10], we begin the proof of the theorem with the transformation

$$\Psi = g^{-1}\Phi, g = \Phi|_{\lambda=\beta}, \tag{20}$$

where Ψ is a solution to the system of the corresponding the GHFE (11), Φ - solution of system, corresponding to the HSE (2)-(3), and $g(x,t)$ - an arbitrary 2x2 matrix function that is a solution to the system (8)-(9) for $\lambda = \beta$.

The derivative of the vector function Ψ with respect to x is equal to

$$\Psi_x = (g^{-1}\Phi)_x = (\lambda - \beta) m g^{-1} \Sigma g = U_1 \Psi . \tag{21}$$

Carrying out a similar calculation for the derivatives with respect to t, we obtain

$$\Psi_t = \left(g^{-1}\Phi\right)_t = (\beta - \lambda) u m g^{-1} \Sigma g + \frac{(\beta - \lambda)}{2\lambda\beta} g^{-1} \Upsilon g = V_1 \Psi , \tag{22}$$

where

$$\Sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \Upsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let us introduce the notation

$$A = g^{-1} \sigma_3 g , \tag{23}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is Pauli matrix.

After some calculations, we get

$$\left[A, A_x\right] = 4\beta m g^{-1} \Sigma g + 4 g^{-1} \Upsilon g , \tag{24}$$

$$\left[A, A_t\right] = \left(\frac{u_{xx}}{2\beta m} + \frac{1}{2\beta} - u \right) \left[A, A_x\right] - \frac{2u_{xx}}{\beta m} g^{-1} \Upsilon g , \tag{25}$$

$$g^{-1} \Sigma g = \frac{1}{4\beta m} \left[A, A_x\right] - \frac{1}{\beta m} g^{-1} \Upsilon g , \tag{26}$$

and

$$g^{-1} \Upsilon g = \frac{1}{4} \left(1 + \frac{m}{u_{xx}} \right) \left[A, A_x\right] - \frac{\beta m}{2u_{xx}} \left[A, u A_x + A_t\right]. \tag{27}$$

Substituting (24) - (27) into (21), (22), we obtain the Lax representation for GHFE (18), (19). Now it is easy to verify that the zero curvature condition

$$U_{1t} - V_{1x} + [U_1, V_1] = 0 , \tag{28}$$

with allowance for (16) and (17) is equivalent to the GHFE. Proposition 2 is proved.

Corollary. If the functions $u(x,t)$ and $m(x,t)$ are solutions of the HSE (2)-(3), then their connections with the solution A for the GHFE (13) are expressed as (14).

Conclusion. In this paper, a matrix form of the Lax representation of the HSE in symmetric space $su(n+1)/s(u(1) \oplus u(n))$ for the case $n = 2$ was proposed. The Lax representation of this type expands the possibilities of studying the equation under consideration. In particular, using the matrix form of the Lax representation for the HSE, we have established the gauge equivalence of this equation to the GHFE and presented the relationship between their solutions.

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ХАНТЕР-САКСТОН ТЕҢДЕУІ МЕН ГЕЙЗЕНБЕРГ ФЕРРОМАГНЕТИГІНІҢ ЖАЛПЫЛАНҒАН ТЕҢДЕУІНІҢ ЭКВИВАЛЕНТТІЛІГІ

Аннотация. Интегралданатын жүйелер қазіргі математикада, теориялық және математикалық физикада маңызды рөл атқарады. Нақты шешімдері бар интегралданатын теңдеулер мен кейбір арнайы шешімдердің көрінісі оның әр түрлі қасиеттерін талдауда маңызды шарттарды қамтамасыз ете алады. Мұндай интегралданатын жүйелер тобына Хантер-Сакстон теңдеуі жатады. Бұл теңдеудің негізінде жатқан көлемді әрі қызықты математикалық теория белсенді түрде математикалық және физикалық зерттеулерге ие болып келеді. Хантер-Сакстон теңдеуі (ХСТ) – бәрімізге белгілі Камасса-Холм теңдеуінің жоғары жиілікті шегі. ХСТ-нің физикалық мағынасы әлсіз сызықты емес бағдарланған толқындардың сұйық кристалл директорының массивті нематикалық өрісінде таралуы болып табылады.

Бұл мақалада $su(n+1)/s(u(1)\oplus u(n))$ симметриялық кеңістігінде, $n = 2$ жағдайы үшін ХСТ -нің Лакс көрінісінің матрицалық түрін ұсынамыз. 1968 жылы Питер Лакс енгізген Лакс жұптары интегралданатын эволюциялық дифференциалдық теңдеулердің сақталатын шамаларын табуға септігін тигізеді. Лакс көрінісі қарастыратын теңдеудің мүмкіншіліктерін арттырады. Мысалы, берілген жұмыста біз Гейзенберг ферромагнетигінің жалпыланған теңдеуімен (ГФЖТ) ХСТ-нің калибрлік эквиваленттілігін орнату үшін, ХСТ-нің матрицалық түрдегі Лакс көрінісін пайдалана-мыз. Әйгілі Гейзенбергтің ферромагнетик теңдеуі (ГФТ) кері шашырау түрлендіруі арқылы интегралданатын классикалық теңдеулердің бірі болып табылады. Бұл мақалада біз оның жалпылауын қарастырамыз. Сонымен қатар, ХСТ мен ГФЖТ шешімдерінің арасындағы байла-нысты көрсетеміз.

Түйін сөздер: интегралданатын жүйелер, Хантер-Сакстон теңдеуі, Гейзенберг ферромагнети-гінің жалпыланған теңдеуі, Лакс көрінісі, калибрлік эквиваленттілік.

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ЭКВИВАЛЕНТНОСТЬ УРАВНЕНИЯ ХАНТЕРА-САКСОНА И ОБОБЩЕННОГО УРАВНЕНИЯ ФЕРРОМАГНЕТИКА ГЕЙЗЕНБЕРГА

Аннотация. Интегрируемые системы играют важную роль в современной математике, теоретической и математической физике. Отображение интегрируемых уравнений с точными решениями и некоторых специальных решений может обеспечить важные условия для анализа его различных свойств. К семейству таких интегрируемых систем принадлежит уравнение Хантера-Сакстона. Обширная и интересная математическая теория, которая лежит в основе уравнения Хантера-Сакстона вызывает активные математические и физические исследования. Уравнение Хантера-Сакстона (УХС) – это высокочастотный предел известного уравнения Камассы-Холма. Физической интерпретацией УХС является распространение слабонелинейных ориентационных волн в массивном нематическом поле директора жидкого кристалла.

В этой статье мы предлагаем матричную форму представления Лакса УХС в симметричном пространстве $su(n+1)/s(u(1)\oplus u(n))$ для случая $n = 2$. Пары Лакса, введенные в 1968 году Питером Лаксом, являются инструментом для нахождения сохраняющихся величин интегрируемых эволюционных дифференциальных уравнений. Представление Лакса расширяет возможности рассматриваемого уравнения. Например, в данной работе мы будем использовать матричное представление Лакса для УХС, чтобы установить калибровочную эквивалентность этого уравнения с обобщенным уравнением ферромагнетика Гейзенберга (ОУФГ). Знаменитое уравнение ферромагнетика Гейзенберга (УФГ) является одним из классических уравнений, интегрируемых посредством обратного преобразования рассеяния. В этой статье мы рассмотрим его обобщение, а также будет представлена связь между решениями УХС и ОУФГ.

Ключевые слова: интегрируемые системы, уравнение Хантера-Сакстона, обобщенное уравнение ферромагнетика Гейзенберга, представление Лакса, калибровочная эквивалентность.

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