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3D MODELING OF HEAT TRANSFER PROCESSES IN THE COMBUSTION CHAMBER BOILER OF THERMAL POWER PLANTS

Abstract. In the present work, a computer-aided 3D modeling method was used to conduct a comprehensive study of heat and mass transfer processes in turbulent flows of high-temperature reactive media in real geometry. Numerical computations of the thermal processes and aerodynamic characteristics of the flow were made for the combustion chamber of the BKZ-75 boiler at Shakhtinskaya thermal power plant for combustion of high-ash fuel. Using the methods of computer 3D-modeling, we took into account a great number of phenomena and factors affecting the real technological processes in the combustion chambers of industrial facilities. The aerodynamic picture of the studied combustion chamber was obtained, the temperature fields and energy distributions released due to chemical reactions were constructed, and the values of radiation heat fluxes to the main heat-receiving surfaces of the combustion chamber were determined. The results of numerical calculations can be used to design new and modernize existing combustion chambers of industrial boilers working on solid fuel, as they are based on the most advanced physical and mathematical models in this area. The use of modern technologies for 3D numerical computations of solid fuel combustion in the combustion chambers of thermal power plants, will allow us to describe in detail the fields of velocity, temperature, pressure and concentrations of all combustion products and, above all, harmful substances and other characteristics of the coal combustion process throughout the combustion space and at the outlet of the combustion chamber.
Key words. Combustion, modelling, thermal power plant, high ash coal.

Introduction

The study of combustion at the level of mathematical modeling is an intermediate link between research conducted at the level of engineering practice and fundamental science [1-3]. It becomes necessary to create new models that will allow us to make more accurate calculations of the fields of velocity, temperature and concentration of the main components of fuel and combustion products in systems such as combustion chambers, various combustion devices, etc. Limitations of theoretical methods and complexity of experimental investigations predetermined a significant role of numerical methods and numerical computations in the study of complex flows of reacting liquids [4-8]. Though, in most cases, mathematical studies are carried out in one- and two-dimensional approximations, and only in rare cases three-dimensional models are used [9-12], moreover, numerical computations are made with constraints in the computational domain. The first results of three-dimensional modeling of heat and mass transfer processes in the combustion chambers of real power facilities of the Republic of Kazakhstan are presented in [13-16]. So the study of the heat transfer processes in furnaces becomes particularly important.

Lately, whole complexes of programs are created, allowing to carry out numerical studies of the most complex phenomena, which include processes of convective heat and mass transfer in high-temperature and chemically reactive flows in the presence of fast-flowing physical and chemical transformations of substances. For this purpose, commercial packages of universal programs that use the latest achievements of computer technology, mathematics, combustion, heat and mass transfer have been developed and applied [17-19].

Modeling of coal combustion

In the present work, physical-mathematical and chemical models were used to study heat and mass transfer in high-temperature environments [20-22]. These models include a system of three-dimensional Navies - Stokes equations and heat and mass transfer equations, considering the source terms determined by the chemical kinetics of the process, nonlinear effects of thermal radiation, interfacial interaction, and multi-stage chemical reactions. The basic equations used to solve the problem are:

The equation for turbulent kinetic energy dissipation ε:

$$\frac{\partial(p\varepsilon)}{\partial t} = -\frac{\partial(pu_j\varepsilon)}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\frac{\mu_{eff}}{\sigma_\varepsilon}\frac{\partial\varepsilon}{\partial x_j}\right] + C_{\varepsilon,1} * \frac{\varepsilon}{k} * P - C_{\varepsilon,2} * \frac{\varepsilon^2}{k} * p \qquad (1)$$

where $p\varepsilon$ is transformation of kinetic energy pulsations into internal energy (dissipation); $\sigma_k, \sigma_\varepsilon$ are turbulent Prandtl numbers.

The basic equations used in this work can be written in generalized form as follows:

$$\frac{\partial(p\phi)}{\partial t} = -\frac{\partial(pu_1\phi)}{\partial x_1} - \frac{\partial(pu_2\phi)}{\partial x_2} - \frac{\partial(pu_3\phi)}{\partial x_3} + \frac{\partial}{\partial x_1}\left[\Gamma_\phi\frac{\partial\phi}{\partial x_1}\right] +$$

$$\frac{\partial}{\partial x_2}\left[\Gamma_\phi\frac{\partial\phi}{\partial x_2}\right] + \frac{\partial}{\partial x_3}\left[\Gamma_\phi\frac{\partial\phi}{\partial x_3}\right] + S_\phi \qquad (2)$$

where ϕ is a transport variable; S_ϕ is the source term determined by the chemical kinetics of the process, nonlinear effects of thermal radiation, interphase interaction and multi-stage chemical reactions. The above system of equations is solved numerically using the control volume method described in detail in [21-24] and used in numerical computations of high-ash coal combustion in Kazakhstan's thermal power plants.

To solve the problem, the mathematical model should include specific initial and boundary conditions for desired functions (velocity, temperature, concentration of the mixture components, etc.) corresponding to the geometry of the selected combustion chamber and the real technological process of fuel combustion at TPPs.

Initial conditions: $u = 0,\ v = 0,\ w = 0,\ P = 0$, at $t = 0$.

The boundary conditions are set on the free surfaces, which are the burners, the exit from the furnace chamber of the boiler and the plane of symmetry.

Input: u_i are speed values, c_β is the initial concentration of each component, the enthalpy h is determined by the input flow temperature from the following relation:

$$C_P = \frac{\partial h}{\partial T} \qquad (3)$$

where T is the temperature at the inlet (experiment or calculation).

Output: $\left.\frac{\partial u_i}{\partial x_i}\right|_{normalA} = 0$, $\left.\frac{\partial h}{\partial x_i}\right|_{normalA} = 0$, $\left.\frac{\partial c_\beta}{\partial x_i}\right|_{normalA} = 0$ are derivatives of velocity, enthalpy and concentration of components normal to the output plane.

In the plane of symmetry: $u_i|_{normalS} = 0$ is the velocity normal to the plane of symmetry, $\left.\frac{\partial u_i}{\partial x_i}\right|_{normalS} = 0$, $\left.\frac{\partial h}{\partial x_i}\right|_{normalS} = 0$, are the derivatives of velocity and enthalpy normal to the plane of symmetry, $\left.\frac{\partial h}{\partial x_i}\right|_{taS} = 0$ is the derivative of the enthalpy tangential to the plane of symmetry, $\left.\frac{\partial c_\beta}{\partial x_i}\right|_{normalS} = 0$ is the derivative of component concentrations normal to the plane of symmetry.

On the solid surface: $u_i|_{normalB} = 0$, $\left.\frac{\partial u_i}{\partial x_i}\right|_{normalB} = 0$, $u_i|_{taB} = 0$, $\partial p|_{boundary} = 0$ is the correction for pressure on the border of the solid surface, $\left.\frac{\partial c_\beta}{\partial x_i}\right|_{normalB} = 0$.

The boundary conditions for the temperature on the wall are determined by the convective heat flux $q_w = \alpha(T_{Steam} - T_{Surf})$. In case of variable temperature of the wall of the combustion chamber, the heat flux can be calculated by the formula:

$$\dot{q} = \underbrace{\sigma(T_{FG} - T_{sufr})}_{convection} + \underbrace{C_{12}(T_{FG}^4 - T_{Surf}^4)}_{radiation} \quad (4)$$

where $C_{12} = \varepsilon_{12}\sigma$, T_{FG} is the temperature of the flue gases, T_{Surf} is the surface temperature of the chamber wall, α is the coefficient of heat transfer by convection, W/(m^{2}K), ε_{12} is the emissivity wall, σ is the Boltzmann constant, W/(m^{2}K^4).

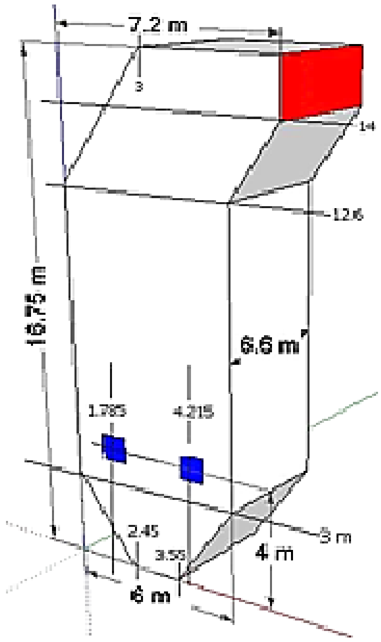
In this work the radiant heat exchange was calculated using the flux model described in [25-26]. The modeling method was developed by Lockwood, Shah [25] and De Marco, Lockwood [26].

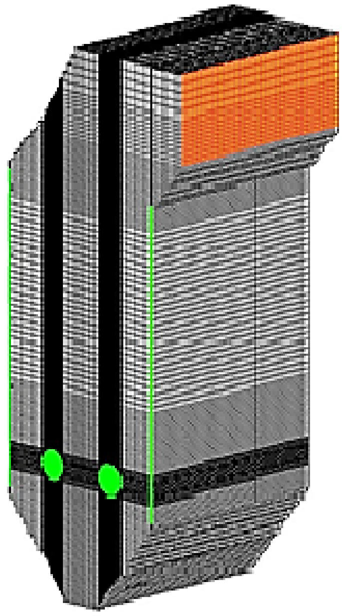
Results of numerical computations

As the object of research has been chosen the boiler BKZ-75 (Fig. 2) located at Shakhtinskaya TPP (Kazakhstan) [27-30]. For numerical simulation, the entire computational domain is divided by a difference grid into discrete points or volumes (Fig. 2b). The resulting finite-difference grid has the resolution of 110×61×150 or 1 006 500 control volumes [31-36].

This paper presents the results of calculations give changes in the velocity and radiant vectors in the sections of the combustion chamber and the temperature profile shown in Figs. 3-5.

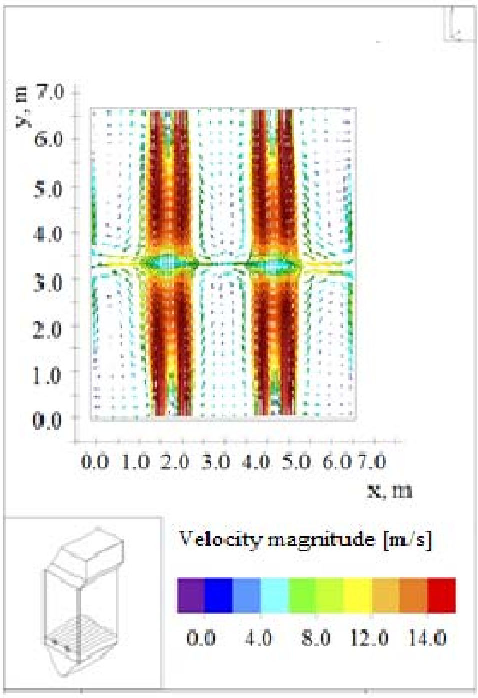
Fig. 3 illustrations the three-dimensional distribution of the full velocity vector in the volume of the combustion chamber. An analysis of Fig. 3a shows that the flow of the air mixture with combustion products has a vortex character in the region of the burners and in the lower part of the combustion chamber. In the center of the combustion chamber, the flux forms several vortices with the presence of a return flow up and down the space of the combustion chamber (Fig. 3b).

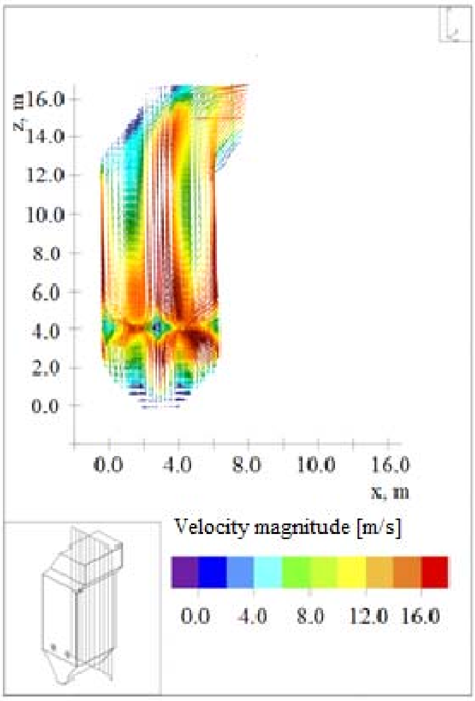




a) b)

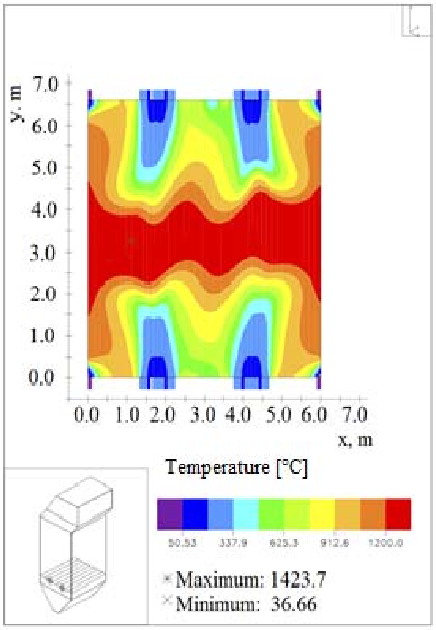
Figure 2 - General view of the BKZ-75 boiler at the Shakhtinskaya TPP a)
and its discretization for control volumes b)

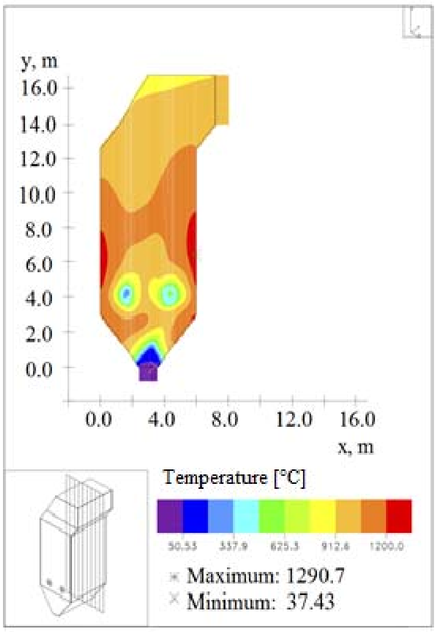




a) cross section (z= 4.0 m), b) longitudinal section (y=3.3 m)

Figure 3 - Velocity distribution in the combustion chamber





a) cross section (z= 4.0 m) b) longitudinal section (y=3.3 m)

Figure 4 - Temperature distribution in the combustion chamber

Fig. 4 shows the temperature distributions characterizing thermal behavior of a pulverized coal flow in the studied combustion chamber. It can be noted that the temperature reaches its maximum values in the region close to the location of the burners, because here, due to the vortex character of the flow, a maximum convective transfer is observed and, as a result, the residence time of coal particles increases, which leads to an increase in temperature in this zone.

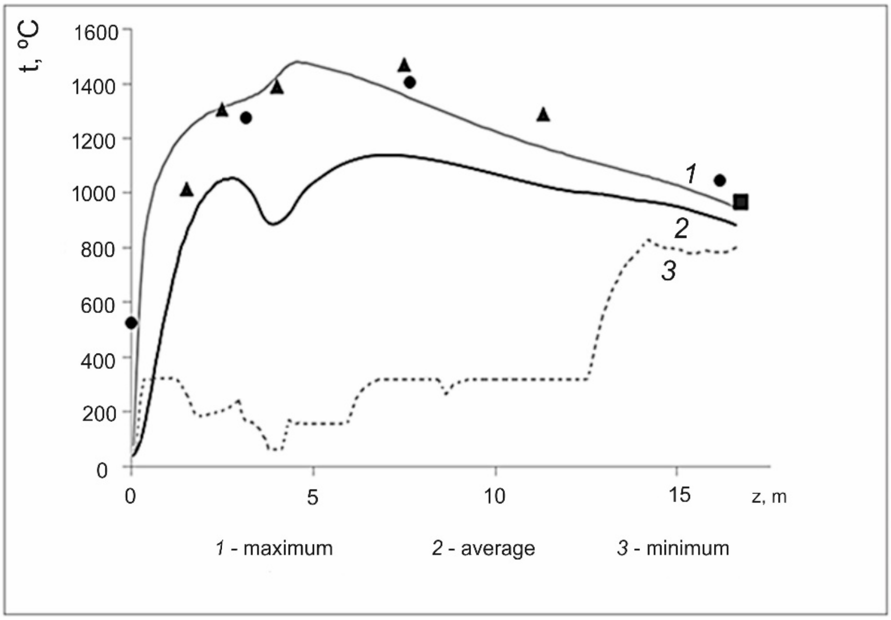
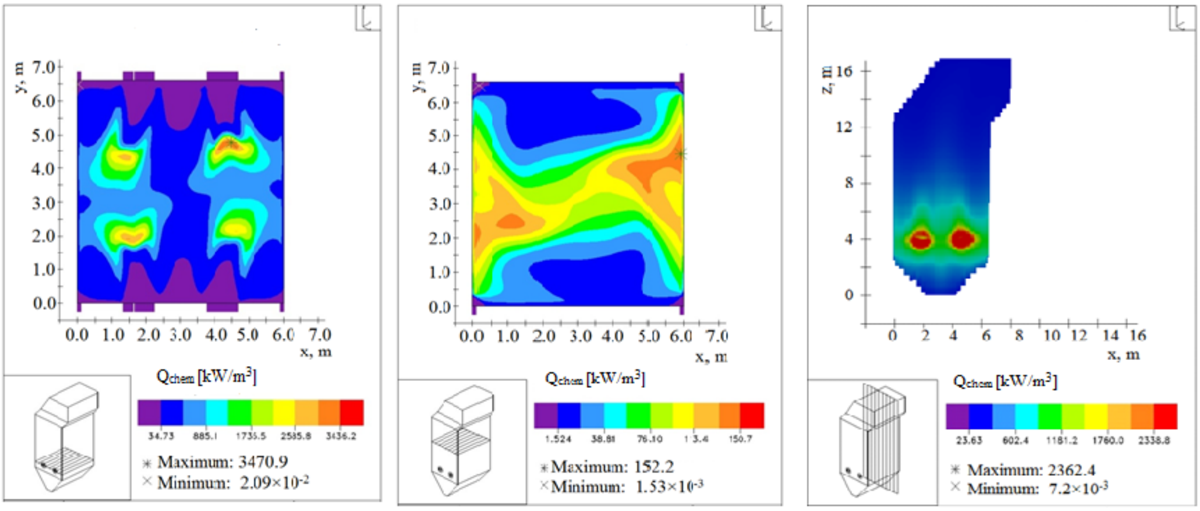


Figure 5 -Temperature distribution along the height of the furnace chamber

Lines correspond to numerical computations; ■ are theoretical values obtained by the method of thermal calculation (CBTI – Central Boiler-and-Turbine Institute) [37]; ▲, ● are the experimental data obtained at the thermal power plant [38-39]

Analysis of Fig. 5 shows that the results of numerical simulation of temperature dependence on the height of the combustion chamber agree with enough accuracy with the theoretical values obtained by the method of thermal calculations suggested by CBTI (Central Boiler-Turbine Institute) [37] and the data obtained directly at TPP [38-39]. This enables us to assess the reliability of the obtained results and the applicability of the physical, mathematical and numerical model to further study of thermal characteristics of the BKZ-75.



a) the burner section b) the central section c) the longitudinal section

Figure 6 - Energy distribution in chemical reactions

As a result of numerical simulation, the energy distribution of chemical reactions in the main sections of the furnace space, the radiation vector profiles in the central sections of the furnace and the distribution of the radiant energy flow to the walls of the combustion chamber were obtained.

The Fig. 6 illustrations that chemical reactions with the highest heat release occur in the fuel and oxidizer supply, i.e. near the installation of burners. In this area mixing of combustible substances and oxygen in the air reaches its maximum level due to intensive mixing, turbulent pulsations and a vortex flow character. This in turn contributes to an increase in the rate of the chemical reaction of carbon oxidation with the release of the maximum amount of energy (Q_{chem}= 3470.9 kW/m^3).

Conclusion

Based on the results of us study, the following conclusions can be drawn:

- The temperature reaches its maximum values in the area close to the location of burners as here, due to the vortex character of the flow, the maximum convective transfer is observed and as a result, the residence time of coal particles increases, which leads to an increase in the temperature in this zone.
- The energy released by chemical reactions reaches its maximum value 3470.9 kW/m^3 in the section of burners. In central part of the combustion chamber this value is 152.2 kW/m^3.
- The physical and mathematical model used in the numerical calculations adequately describes burning of high-ash coal in the combustion chamber of the BKZ 75-39 boiler at Shakhtinskaya TPP, Kazakhstan. The obtained results are in good agreement with the experimental data, which were obtained specially at the thermal power station.

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ЖЫЛУ ЭЛЕКТР СТАНЦИЯЛАРЫНЫҢ ЖАНУ ҚАЗАНЫНДА ЖЫЛУ АЛМАСУ ПРОЦЕСТЕРІН 3D МОДЕЛЬДЕУ

Аннотация. Бұл жұмыста нақты геометрия аймақтарында (ЖЭС, ЖЭО) жоғары температурадағы әсер ететін ортадағы турбулентті ағындарында жылу және масса тасымалының процестерін жан-жақты зерттеу үшін компьютерлік 3D модельдеу әдісі пайдаланылды. Шахтинск ЖЭО зауытындағы БКЗ-75 қазандықтың жану камерасында жылу процестерін және ағынның аэродинамикалық сипаттамаларын зерттеуге арналған есептік эксперименттер жоғары күлді жанармайдың жануы кезінде жүргізілді. Компьютерлік 3D модельдеу әдістерін пайдаланған кезде өндірістік объектілердің жану камераларында нақты технологиялық үрдістер ағынынна әсер ететін құбылыстар мен факторлардың ең көп саны ескерілді. Зерттелетін жану камерасының аэродинамикалық сүлбесі ұсынылған, химиялық реакциялар арқылы пайда болатын температура өрістері мен энергия бөлу, сондай-ақ жану камерасының негізгі жылу алатын беттеріне радиациялық жылу ағындарының мәндері алынады. Орындалған есептеу эксперименттерінің нәтижелері қатты отынмен жұмыс істейтін өнеркәсіптік қазандықтардың жану камераларын жаңа және жобалау кезінде, физикалық және математикалық модельдер осы саладағы ғылымның даму деңгейіне арналған ең толық, заманауи және оңтайлы болып табылады мүмкін. Жану камерасында жылу электр станцияларының жану камераларында қатты отынды жағу бойынша 3D есептеу эксперименттеріне арналған заманауи технологияларды қолдану барлық жану өнімдерінің, сонымен бірге жану аймағындағы жанғыш көмір процесінің зиянды заттар мен басқа да сипаттамаларының жылдамдығын, температурасын, қысымын және концентрациясын егжей-тегжейлі сипаттауға мүмкіндік береді.

Түйін сөздер. Жылу масса алмасу, жану, қатты отын, плазмалық активация, ародинамикалық ағыс, концентрация және температура өрісі, зиянды заттардың қалдықтары.

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3D-МОДЕЛИРОВАНИЕ ПРОЦЕССОВ ТЕПЛООБМЕНА В КАМЕРЕ СГОРАНИЯ КОТЛА ТЕПЛОВЫХ ЭЛЕКТРИЧЕСКИХ СТАНЦИЙ

Аннотация. В настоящей работе методами компьютерного 3D-моделирования проведено комплексное исследование процессов тепломассопереноса в турбулентных течениях высокотемпературных реагирующих сред в областях реальной геометрии (ТЭС, ТЭЦ). Вычислительные эксперименты по исследованию тепловых процессов и аэродинамических характеристик течения проведены в топочной камере котла БКЗ-75 Шахтинской ТЭЦ при сгорании в ней высокозольного энергетического топлива. При использовании методов компьютерного 3D-моделирования учтено наибольшее количество явлений и факторов, влияющих на протекание реальных технологических процессов в камерах сгорания промышленных объектов. Представлена аэродинамическая картина исследуемой топочной камеры, построены температурные поля и распределения энергии, выделяющейся за счет химических реакций, а также получены значения радиационных тепловых потоков на основные тепловоспринимающие поверхности камеры сгорания. Результаты проведенных вычислительных экспериментов, могут быть использованы при проектировании новых и доработке существующих топочных камер промышленных котлов, использующих твердое топливо, поскольку используемые физико-математические модели являются наиболее полными, современными и оптимальными для данного уровня развития науки в этой области. Применение современных технологий для проведения 3D-вычислительных экспериментов по сжиганию твердого топлива в топочных камерах ТЭС, позволит подробно описать поля скорости, температуры, давления и концентраций всех продуктов сжигания и прежде всего вредных веществ и других характеристик процесса сжигания угля по всему топочному пространству и на выходе из топочной камеры.

Ключевые слова. Тепломассоперенос, горение, твердое топливо, плазменная активация, аэродинамика течения, концентрационные и температурные поля, выбросы вредных веществ.

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REFERENCES

[1] Baktybekov KS, Syzdykov AB, Ospanova Sh, etc (**2018**) Investigation of the different Reynolds numbers influence on the atomization and combustion processes of liquid fuel. Bulgarian Chemical Communications, 50(G):68–77.

[2] Askarova A, Bolegenova S, Shortanbaeva ZhK, etc (**2017**) Numerical modeling of burning pulverized coal in the combustion chamber of the boiler PK 39. News of the national academy of sciences of the Republic of Kazakhstan-series physico-mathematical, 2(312):58-63.

[3] Nugymanova A, Yergaliyeva A, Shortanbayeva Zh, etc (**2017**) Simulation of the aerodynamics and combustion of a turbulent pulverized-coal flame. Proceedings of 4th International Conference on Mathematics and Computers in Sciences and in Industry (MCSI 2017). Corfu Island, Greece. P.92-97. DOI: 10.1109/MCSI.2017.23.

[4] Bolegenova SA, Maxutkhanova AM, Beisenov KhI, etc (**2017**) A computational experiment for studying the combustion of thermochemically-gasified coal in the combustion chamber of the boiler BKZ-160. News of the national academy of sciences of the Republic of Kazakhstan-series physico-mathematical, 2(312):78-80 DOI: 10.1515/eng-2018-0020.

[5] Askarova AS, Bekmuhamet A, etc (**2012**) Mathematical simulation of pulverized coal in combustion chamber. Procedia Engineering, 42:1150-1156. DOI: 10.1016/j.proeng.2012.07.507.

[6] Safarik P, Beketayeva MT, etc (**2018**) Modern computing experiments on pulverized coal combustion processes in boiler furnaces. News of the national academy of sciences of the Republic of Kazakhstan-series Physico-mathematical, 6(322): 5-14 DOI: 10.32014/2018.2518-1726.11.

[7] Beketayeva MT, Ergalieva A, Gabitova Z, etc (**2016**) Computational modeling of heat and mass transfer processes in combustion chamber at power plant of Kazakhstan. Proceedings MATEC Web of Conferences, 76:UNSP06001 DOI:10.1051/matecconf/20167606001.

[8] Mazhrenova NR, Bolegenova SA, Mamedova MR, etc **(2019)** Computational experiments for research of flow aerodynamics and turbulent characteristics of solid fuel combustion process. News of the national academy of sciences of the Republic of Kazakhstan-series physico-mathematical, 2(324):46–52. https://orcid.org/10.32014/2019.2518-1726.11

[9] Bolegenova SA, Beketayeva MT, Askarova A, etc (**2018**) Modeling of heat and mass transfer in high-temperature reacting flows with combustion. Journal High Temperature, 56(5):738-745. DOI: 10.1134/S0018151X1805005X.

[10] Aldiyarova AN, Boranbayeva AE, Maksutkhanova AM, etc (**2016**) Mathematical modeling of heat and mass transfer in the presence of physical-chemical processes. Proceeding of 15th international scientific conference on renewable energy and Innovate Technologies, Tech Coll Smolyan, Smolyan, Bulgaria, Bulgarian Chemical Communications, 48(E):272-277.

[11] Bekmukhamet A, Beketayeva MT, Gabitova ZKh, etc (**2015**) Computational method for investigation of solid fuel combustion in combustion chambers of a heat power plant. Journal High Temperature, 5(5):751-757. DOI: 10.1134/S0018151X15040021.

[12] Safarik P, Maximov VYu, Bolegenova SA, etc **(2019)** 3D modeling of combustion thermochemical activated fuel. News of the national academy of sciences of the Republic of Kazakhstan-series physico-mathematical, 2(324):9–16. https://orcid.org/10.32014/2019.2518-1726.7

[13] Boranbayeva A, Manatbayev R, Maxutkhanova A, etc (**2016**) Application of 3D modelling for solving the problem of combustion coal-dust flame. Proceeding of 15th international scientific conference on renewable energy and Innovate Technologies, Tech Coll Smolyan, Smolyan, Bulgaria, Bulgarian Chemical Communications, 48(E):236-241.

[14] Baizhuma ZhT, Yergalieva AB, Manatbayev RK, etc (**2018**) The use of a new "clean" technology for burning low-grade coal in on boilers of Kazakhstan TPPs. Bulgarian Chemical Communications, 50(G):53 – 60.

[15] Nagibin, AO, Messerle VE, etc (**2010**) Pulverized coal torch combustion in a furnace with plasma-coal system. Journal Thermophysics and Aeromechanics, 17(3):435-444. DOI: 10.1134/S0869864310030145

[16] Messerle VE, Ustimenko AB, Maksimov VYu, etc (**2014**) Numerical simulation of the coal combustion process initiated by a plasma source. Journal Thermophysics and Aeromechanics, 21(6):747-754. DOI: 10.1134/S0869864314060092.

[17] Messerle VE, Karpenko EI, Ustimenko AB, etc (**2006**) Plasma enhancement of combustion of solid fuels. Journal of High Energy Chemistry, 40(2):111-118. DOI: 10.1134/S0018143906020081.

[18] Ospanova S, Bolegenova S, Ergalieva A, etc (**2016**) 3D modeling of heat and mass transfer during combustion of solid fuel in BKZ-420-140-7C combustion chamber of Kazakhstan. Journal of Applied Fluid Mechanics, 9(2):699-709.

[19] Buchmann M, Askarowa A (**1997**) Structure of the flame of fluidized-bed burners and combustion processes of high-ash coal. Proceedings of 18th Dutch-German Conference on Flames, VDI Berichte, 1313:241-244.

[20] Manatbayev RK, Beketayeva MT, Yergaliyeva AB, etc (**2016**) CFD study of harmful substances production in coal-fired power plant of Kazakhstan. Proceedings of 15th International Scientific Conference on Renewable Energy and Innovative Technologies: Tech Coll Smolyan, Smolyan, Bulgaria, Bulgarian Chemical Communications, 48(E2):260-265.

[21] Müller H. **(1992)** Numerische Berechnung Dreidimensionaler Turbulenter Strömungen in Dampferzeugern Mit Wärmeübergang und Chemischen Reaktionen am Beispiel des SNCR–Verfahrens und der Kohleverbrennung: Fortschritt–Berichte VDI-Verlag, 6(268):158 (in German).

[22] Incropera FP, DeWitt DP. **(1981)** Fundamentals of Heat and Mass Transfer. Second Edition. John Wiley& Sons. New York.

[23] Chernetsky MYu., Kuznetsov VA., etc **(2016)** Comparative Analysis of the Effect of Turbulence Models on the Description of the Processes of Burning Coal Dust in the Presence of Flow Swirling. Journal Thermal Physics and Aeromechanics, 23(4):615 - 626

[24] Askarova A, Chtab-Desportes A, Voloshina I, etc (**2010**) Stochastic simulation of the spray formation assisted by a high pressure. AIP Conference Proceedings, 1207:66-73

[25] Lockwood F, Shah N. **(1976)** An Improved Flux Model for Calculation of Radiation Heat Transfer in Combustion Chambers. Proceedings of the ASME-AIChE Heat transfer Conference: ASME-Paper, Salt Lake City, 2-7.

[26] De Marco A, Lockwood F **(1975)** A New Flux Model for the Calculation of Radiation Furnaces. Italian Flame Days, Sanremo, 1-13.

[27] Gabitova ZKh, Messerle VE, Ustimenko AB, etc (2015) Numerical simulation of pulverized coal combustion in a power boiler furnace. Journal High Temperature, 53(3):445-452. DOI: 10.1134/S0018151X15020030.

[28] Vockrodt S, Leithner R, Schiller A, (**1999**) Firing technique measures for increased efficiency and minimization of toxic emissions in Kasakh coal firing. Proceedings of 19th German Conference on Flames, Dresden, Germany VDI Gesell Energietechn; Verein Deutsch Ingn.Combustion and incineration, 1492:93-97. (in Eng).

[29] Karpenko EI, Karpenko YeE, Ustimenko AB, etc (**2007**) Mathematical modeling of the processes of solid fuel ignition and combustion at combustors of the power boilers. Proceedings of the 7th International Fall Seminar on Propellants, Explosives and Pyrotechnics, Xian, 7:672-683.

[30] Askarova A, Bolegenova SA, Ospanova ShS, etc (**2017**) Investigation of aerodynamics and heat and mass transfer in the combustion chambers of the boilers PK-39 and BKZ-160. News of the national academy of sciences of the Republic of Kazakhstan-series physico-mathematical, 2(322):5-14.

[31] Berezovskaya I, Ospanova S., Shortanbayeva Zh, etc (**2016**) 3D modelling of heat and mass transfer processes during the combustion of liquid fuel, Proceedings of 15th International Scientific Conference on Renewable Energy and Innovative Technologies, Tech Coll Smolyan, Smolyan, Bulgaria, Bulgarian Chemical Communications, 48(E):229-235.

[32] Maximov VYu, Messerle VE, Ustimenko AB, etc (**2016**) Reduction of noxious substance emissions at the pulverized fuel combustion in the combustor of the BKZ-160 boiler of the Almaty heat electro power station using the “Overfire Air” technology. Journal Thermophysics and Aeromechanics, 23(1):125-134. DOI: 10.1134/S0869864316010133.

[33] Ospanova ShS., mailto:maximov.v@mail.ruBekmuhamet A, etc (**2012**) Numerical research of aerodynamic characteristics of combustion chamber BKZ-75 mining thermal power station. Procedia Engineering, 42:1250-1259. DOI: 10.1016/j.proeng.2012.07.517.

[34] Bekmukhamet A, Beketayeva M, Gabitova Z, etc (**2014**) Control harmful emissions concentration into the atmosphere of megacities of Kazakhstan Republic. International conference on Future Information Engineering, Beijing, Peoples China, 10:252-258. DOI: 10.1016/j.ieri.2014.09.085.

[35] Nugmanova A, Berezovskaya I, Gabitova Z etc (**2017**) Experimental study of complex combustion progress in higher education. Proceedings of 2nd International conference on Education, Management and Systems, Beijing, Peoples China, 377-381.

[36] Ospanova ShS, Manatbayev RK, Beketayeva MT, etc (**2018**) The computational study of heat and mass transfer processes at combustion of pulverized kazakh coal in real conditions of energy objects. Journal Bulgarian Chemical Communications, 50(G):61- 67.

[37] Mitor VV, Kuznecov NV, Dubovsky IE, etc. **(1973)** Thermal Calculation of Boiler Units. Normative Method. Moscow. Energy, 297.

[38] Alijarov BK, Alijarova MB (**2012**) Szhiganie kazahstanskih uglej na TJeS i na krupnyh kotel'nyh: opyt i perspektivy. RGP Gylym ordasy, Kazakhstan.

[39] Alijarov BK, **(1996)** Soveshenstvovanie szhiganija jekibastuzskogo uglja na teplovyh elektrostancijah, RGP Gylym ordasy, Kazakhstan.

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NUMERICALLY APPROXIMATE METHOD FOR SOLVING OF A CONTROL PROBLEM FOR INTEGRO-DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE

Abstract. A linear boundary value problem with a parameter for integro-differential equations of parabolic type is investigated. Using the spatial variable discretization, the considering problem is approximated by a linear boundary value problem with a parameter for a system of ordinary integro-differential equations.The parameterization method is used for solving the obtained problem. The approximating problem is reduced to an equivalent problem consisting of a special Cauchy problem for the system of Fredholm integro-differential equations, boundary conditions, and continuity conditions of the solution at the partition points. The solution of the Cauchy problem for the system of ordinary differential equations with parameters is constructed using the fundamental matrix of the differential equation. The system of a linear algebraic equations with respect to the parameters are composed by substituting the values of the corresponding points in the boundary condition and the continuity conditions. Numerical method for solving of the problem is suggested, which based on the solving of the constructed system and method of Runge-Kutta 4-th order for solving of the Cauchy problem on the subintervals.

Key words:partial integro-differential equations of parabolic type, problem with parameter, approximation, numericallyapproximate method, algorithm.

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Control problems, which are also called boundary value problems with parameters and the problem of parameter identification for a system of ordinary differential and integro-differential equations with parameters, have been actively investigated in recent decades. Models describing reaction-diffusion processes lead to control problems for integro-differential equations of parabolic type [1-17].Questions of existence, uniqueness and stability of solving problems with parameters are very important for development of numerical methods of identification of parameters of the mathematical models described by integro-differential equations of parabolic type [1-17].

In the present paper, linear problem with a parameter for an integro-differential equation of parabolic type is investigated. By discretizing a spatial variable, the considering problem is approximated by a two-point boundary value problem with parameters for a system of Fredholm integro-differential equations with a degenerate kernel.By introducing additional parameters [18-23] as the values of the desired solution at some points of the interval $[0, T]$, where the problem is considered, the obtained problem is reduced to the equivalent problem consisting of a special Cauchy problem for the system of Fredholm integro-differential equations, boundary conditions, and continuity conditions for the solution at the points of partition.Using the integral equation, that equivalent to the special Cauchy problem for the system of Fredholm integro-differential equation and the property of the degeneracy of kernel of the integral term, we obtained a representation of the solution of the special Cauchy problem using the entered parameters at the assumption of invertibility of a some matrix. Based on this representation, a system of algebraic equations with respect to the parameters is constructed from the boundary conditions and the continuity conditions of the solution. We offer algorithm for solving the linearboundaryvalue problem for the equation with degeneratekernel,anditsnumerical implementation.

We consider a linear boundary value problem with a parameter for an integro-differential equation of parabolic type

$$\frac{\partial u}{\partial t} = a(x,t)\frac{\partial^2 u}{\partial x^2} + c(x,t)u + b(x,t)\mu(x) +$$

$$+\varphi(x,t)\int_0^T \psi(x,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega = (0,\omega)\times(0,T), \tag{1}$$

$$u(x,0) = 0,\ x \in [0,\omega], \tag{2}$$

$$u(x,T) = 0,\ x \in [0,\omega], \tag{3}$$

$$u(0,t) = \tilde{\psi}_1(t),\ u(\omega,t) = \tilde{\psi}_2(t),\ t \in [0,T], \tag{4}$$

where $u(x,t)$issought function, $\mu(x)$isunknown functional parameter, functions$a(x,t) \geq a_0 > 0$, $c(x,t) \leq 0$, $b(x,t)$, $\varphi(x,t)$,$\psi(x,t)$, $f(x,t)$are continuous in $t$and Holder continuous in$x$ onΩ; functions$\tilde{\psi}_1(t)$, $\tilde{\psi}_2(t)$are continuous on $[0,T]$. It is assumed that the boundary functions are sufficiently smooth and satisfy the matching conditions.

The solution of the boundary problem (1)-(4) is a pair of functions$(u^*(x,t),\mu^*(x))$,where function $u^*(x,t)$ is continuousonΩ, that has continuouspartial derivatives with respect to x of first order, with respect to t of second order, satisfies the integro-differential equation (1) at $\mu(x) = \mu^*(x)$, $x \in [0,\omega]$,and boundary conditions (2)-(4).

In view of condition (2)-(4), from (1) we obtain two groups of equations for determining $\mu(0)$and$\mu(\omega)$:

$$b(0,0)\mu(0) = \dot{\tilde{\psi}}_1(0) - \varphi(0,0)\int_0^T \psi(0,s)\tilde{\psi}_1(s)ds - f(0,0),$$

$$b(\omega,0)\mu(\omega) = \dot{\tilde{\psi}}_2(0) - \varphi(\omega,0)\int_0^T \psi(\omega,s)\tilde{\psi}_2(s)ds - f(\omega,0),$$

$$b(0,T)\mu(0) = \dot{\tilde{\psi}}_1(T) - \varphi(0,T)\int_0^T \psi(0,s)\tilde{\psi}_1(s)ds - f(0,T),$$

$$b(\omega,T)\mu(\omega) = \dot{\tilde{\psi}}_2(T) - \varphi(\omega,T)\int_0^T \psi(\omega,s)\tilde{\psi}_2(s)ds - f(\omega,T).$$

These relations also are the matching conditions with respect to initial data.

We take $\forall h > 0$ and produce a discretization by x: $x_i = ih, i = \overline{0,P}, Ph = \omega$.

We introduce the notations$u_i(t) = u(ih,t), \mu_i = \mu(ih), a_i(t) = a(ih,t), c_i(t) = c(ih,t), b_i(t) = b(ih,t), \varphi_i(t) = \varphi(ih,t), \psi_i(t) = \psi(ih,t), f_i(t) = f(ih,t), i = \overline{0,P}$.

Problem (1) - (4) is replaced by the following linear boundary value problem with a parameter for an integro-differential equation

$$\frac{du_i}{dt} = a_i(t)\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + c_i(t)u_i + b_i(t)\mu_i +$$

$$+\varphi_i(t)\int_0^T \psi_i(s)u_i(s)ds + f_i(t), \quad i = \overline{1,P-1}, \tag{5}$$

$$u_i(0) = 0,\ i = \overline{0,P}, \tag{6}$$

$$u_i(T) = 0,\ i = \overline{0,P}, \tag{7}$$

$$u_0(t) = \tilde{\psi}_1(t),\ u_P(t) = \tilde{\psi}_2(t),\ t \in [0,T]. \tag{8}$$

The functions$u_0(t)$, $u_P(t)$, and parameters μ_0,μ_P are known.

Problem (5)-(8) will be rewritten in vector-matrix form

$$\frac{du}{dt} = A(t)u + B(t)\mu + \Phi(t)\int_0^T \Psi(s)u(s)ds + F(t), \qquad u,\mu \in R^{P-1}, \qquad t \in (0,T), \tag{9}$$

$$u(0) = 0, \tag{10}$$

$$u(T) = 0, \tag{11}$$

where$u\,(t) = (u_1(t), u_2(t), \dots, u_{P-1}(t))$, $\mu = (\mu_1, \mu_2, \dots, \mu_{P-1})$-unknown function and parameter,

$$A(t) = \begin{pmatrix} -\frac{2a_1(t)}{h^2} + c_1(t) & \frac{a_1(t)}{h^2} & 0 & \cdots & 0 \\ \frac{a_2(t)}{h^2} & -\frac{2a_2(t)}{h^2} + c_2(t) & \frac{a_2(t)}{h^2} & \cdots & 0 \\ 0 & \frac{a_3(t)}{h^2} & -\frac{2a_3(t)}{h^2} + c_3(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\frac{2a_{P-1}(t)}{h^2} + c_{P-1}(t) \end{pmatrix},$$

$$B(t) = diag\{b_1(t), b_2(t), \dots, b_{P-1}(t)\},$$

$$\Phi(t) = diag\{\varphi_1(t), \varphi_2(t), \dots, \varphi_{P-1}(t)\},$$

$$\Psi(s) = diag\{\psi_1(s), \psi_2(s), \dots, \psi_{P-1}(s)\},$$

$$F(t) = \left(\frac{a_1(t)}{h^2}\tilde{\psi}_1(t) + f_1(t), f_2(t), \dots, \frac{a_{P-1}(t)}{h^2}\tilde{\psi}_2(t) + f_{P-1}(t)\right)^{/}.$$

Here $(P-1)\times(P-1)$-matricesA(t), B(t), $\Phi(t)$, $\Psi(s)$and$(P-1)$-vector$F(t)$are continuous on$[0,T]$.

The solution to problem (9) - (11) is a pair$(u^*(t), \mu^*)$, where continuous on $[0,T]$and continuously differentiable on $(0,T)$a function $u^*(t)$satisfies the integro-differential equation (9) at $\mu = \mu^*$and conditions (10), (11).

To solve the problem with parameter (9)-(11), the approach developed in [24-26] is used, based on the algorithms of the parameterization method and numerical methods for solving Cauchy problems.

Scheme of the method.Points$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ are taken and the interval$[0,T)$ is divided into Nsubintervals:$[0,T) = \bigcup_{r=1}^{N}[t_{r-1}, t_r)$, which is denoted by$\Delta_N$[20]. The restriction of the function $u(t)$ to the r −th interval $[t_{r-1}, t_r)$ is denoted by $x_r(t)$, i.e. $u_r(t) = u(t)$for $t \in [t_{r-1}, t_r)$, r=$\overline{1,N}$.

Let$C([0,T], R^{P-1})$be the space of continuous on $[0,T]$functions$u: [0,T] \to R^{P-1}$with norm$\|u\|_1 = \max_{t\in[0,T]}\|u(t)\|$; $C([0,T], \Delta_N, R^{(P-1)N})$ - the space of systems of functions$u[t] = (u_1(t), u_2(t), \dots, u_N(t))$, where $u_r: [t_{r-1}, t_r) \to R^{P-1}$ are continuous on$[t_{r-1}, t_r)$ and have finite left-sided limits$\lim_{t\to t_r - 0} u_r(t)$ for allr=$\overline{1,N}$, with norm$\|u[\cdot]\|_2 = \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)}\|u_r(t)\|$.

We introduce additional parameters $\lambda_r = u_r(t_{r-1})$, r=$\overline{2,N}$, $\lambda_1 = \mu$.Making the substitution $u_1(t) = z_1(t)$, $u_r(t) = z_r(t) + \lambda_r$on every r-th interval $[t_{r-1}, t_r)$, r=$\overline{2,N}$,we obtain multipoint boundary value problem with parameters

$$\frac{dz_1}{dt} = A(t)z_1 + B(t)\lambda_1 + \Phi(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)z_j(s)ds +$$

$$+\Phi(t)\sum_{j=2}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)\lambda_j ds + F(t), \quad t \in [t_0, t_1), \tag{12}$$

$$z_1(t_0) = 0, \tag{13}$$

$$\frac{dz_r}{dt} = A(t)(z_r + \lambda_r) + B(t)\lambda_1 + \Phi(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)z_j(s)ds +$$

$$+\Phi(t)\sum_{j=2}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)\lambda_j ds + F(t), \quad t \in [t_{r-1}, t_r), \tag{14}$$

$$z_r(t_{r-1}) = 0, \ r=\overline{2,N}, \tag{15}$$

$$\lambda_N + \lim_{t\to T-0} z_N(t) = 0, \tag{16}$$

$$\lim_{t\to t_1-0} z_1(t) = \lambda_2, \tag{17}$$

$$\lambda_s + \lim_{t\to t_s-0} z_s(t) = \lambda_{s+1}, \quad s = \overline{2,N-1}. \tag{18}$$

The solution of the problem with parameters (12)-(18) is a pair $(z^*[t], \lambda^*)$where the function$z^*[t] = (z_1^*(t), z_2^*(t), \dots, z_N^*(t)) \in C([0,T], \Delta_N, R^{(P-1)N})$with continuously differentiable components $z_r^*(t)$on$[t_{r-1}, t_r)$and $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_N^*) \in R^{(P-1)N}$, satisfies the integro-differential equation with parameters (12), (14), initial conditions (13), (15), relations (16)-(18) at$\lambda_j = \lambda_j^*, j = \overline{2,N}$.

If the pair$(u^*(t), \mu^*)$ is a solution of problem (9)-(11), then the pair $(z^*[t], \lambda^*)$with elements $z^*[t] = (z_1^*(t), z_2^*(t), \dots, z_N^*(t)) \in C([0,T], \Delta_N, R^{(P-1)N})$, $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_N^*) \in R^{(P-1)N}$, where $\lambda_1^* = \mu^* \in R^{P-1}$, $z_1^*(t) = u_1^*(t)$, $t \in [t_0, t_1)$, $\lambda_r^* = u_r^*(t_{r-1})$, $z_r^*(t) = u_r^*(t) + u_r^*(t_{r-1})$, $t \in [t_{r-1}, t_r)$, $r = \overline{2,N}$, is the solution of problem (12)-(18). Conversely, if a pair $(\tilde{z}[t], \tilde{\lambda})$with elements $\tilde{z}[t] = (\tilde{z}_1(t), \tilde{z}_2(t), \dots, \tilde{z}_N(t)) \in C([0,T], \Delta_N, R^{(P-1)N})$, $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_N) \in R^{(P-1)N}$, is a solution of (12)-(18) , then the pair $(\tilde{u}(t), \tilde{\mu})$defined by the equalities $\tilde{u}(t) = \tilde{z}_1(t)$, $t \in [t_0, t_1)$,$\tilde{u}(t) = \tilde{z}_r(t) + \tilde{\lambda}_r$,$t \in [t_{r-1}, t_r)$, $r = \overline{2,N}$,$\tilde{u}(T) = \lim_{t\to T-0} \tilde{z}_N(t) + \tilde{\lambda}_N$ and $\tilde{\mu} = \tilde{\lambda}_1$, will be the solution of the original boundary value problem with parameter (9) - (11).

Using the fundamental matrix $X_r(t)$of the differential equation$\frac{dx}{dt} = A(t)x$,$t \in [t_{r-1}, t_r)$, $r = \overline{1,N}$, we reduce the solution of a special Cauchy problem for an integro-differential equation with parameters (12)-(15) to an equivalent system of integral equation

$$z_1(t) = X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)\Phi(\tau)\left\{\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)z_j(s)ds + \sum_{j=2}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)\lambda_j ds\right\}d\tau +$$

$$+X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)B(\tau)d\tau\,\lambda_1 + X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)F(\tau)d\tau, t \in [t_0, t_1), \tag{19}$$

$$z_r(t) = X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)\Phi(\tau)\left\{\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)z_j(s)ds + \sum_{j=2}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)\lambda_j ds\right\}d\tau +$$

$$+X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)A(\tau)d\tau\,\lambda_r + X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)B(\tau)d\tau\,\lambda_1 +$$

$$+X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)F(\tau)d\tau, t \in [t_{r-1}, t_r), \ r = \overline{2,N}. \tag{20}$$

Let $\xi = \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)z_j(s)ds$and rewrite the system of integral equations (19), (20) in the form

$$z_1(t) = X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)\Phi(\tau)\left\{\xi + \sum_{j=2}^{N}\int_{t_{j-1}}^{t_j}\Psi(s)\lambda_j ds\right\}d\tau +$$

$$+X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)B(\tau)d\tau\,\lambda_1 + X_1(t)\int_{t_0}^{t} X_1^{-1}(\tau)F(\tau)d\tau, t \in [t_0, t_1), \tag{21}$$

$$z_r(t) = X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau)\Phi(\tau)\left\{\xi + \sum_{j=2}^{N} \int_{t_{j-1}}^{t_j} \Psi(s)\lambda_j ds\right\} d\tau +$$

$$+X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau)A(\tau)d\tau\,\lambda_r + X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau)B(\tau)d\tau\,\lambda_1 +$$

$$+X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau)F(\tau)d\tau\,, t \in [t_{r-1}, t_r),\ \mathrm{r} = \overline{2, N}. \quad (22)$$

Multiplying both parts of (21), (22)by $\Psi(t)$, integrating on$[t_{r-1}, t_r]$, and summing by$r$, we obtain a system of linear algebraic equations with respect to$\xi \in R^{P-1}$

$$\xi = G(\Delta_N)\xi + \sum_{r=1}^{N} V_r(\Delta_N)\,\lambda_r + \mathrm{g}(\mathrm{F}, \Delta_N), \quad (23)$$

with $(P-1) \times (P-1)$-matrices

$$G(\Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \Psi(\tau)X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(s)\Phi(s)dsd\tau,$$

$$V_1(\Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \Psi(\tau)X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(s)\mathrm{B}(s)dsd\tau,$$

$$V_r(\Delta_N) = \int_{t_{r-1}}^{t_r} \Psi(\tau)X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(s)A(s)dsd\tau +$$

$$+\sum_{r=1}^{N}\sum_{j=1}^{N} \int_{t_{r-1}}^{t_r} \Psi(\tau)X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(\tau_1)\Phi(\tau_1)d\tau_1 d\tau \int_{t_{j-1}}^{t_j} \Psi(s)ds\,,\ \mathrm{r} = \overline{2, N}.$$

and$(P-1)$–vectors

$$\mathrm{g}(\mathrm{F}, \Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \Psi(\tau)X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(s)\mathrm{F}(s)dsd\tau.$$

We write the system (23) in the form

$$[I - G(\Delta_N)]\xi = \sum_{r=1}^{N} V_r(\Delta_N)\,\lambda_r + \mathrm{g}(\mathrm{F}, \Delta_N), \quad (24)$$

where$I$is the identity matrix of $(P-1)$dimension.

The special Cauchy problem (12) - (15) is equivalent to the system of integral equations (19) - (20). This system, due to the degeneracy of the kernel, will be equivalent to the system of algebraic equations (23) with respect to$\xi \in R^{P-1}$. The unique solvability of the special Cauchy problem was investigated in [19, 20]. It has been established that with a sufficiently small step$h > 0: Nh = T$ partitioning a segment$[0, T]$ the special Cauchy problem will be unique solvable.

Let the matrix $I - G(\Delta_N)$be invertible, i.e. exists $[I - G(\Delta_N)]^{-1}$. Then, according to (24), the vector $\xi \in R^{P-1}$is determined by the equality

$$\xi = [I - G(\Delta_N)]^{-1} \sum_{r=1}^{N} V_r(\Delta_N)\,\lambda_r + [I - G(\Delta_N)]^{-1}\mathrm{g}(\mathrm{F}, \Delta_N). \quad (25)$$

In (21), (22), instead of ξ substituting the right-hand side of (25), we obtain the representation of the function $z_r(t)$in terms of$\lambda_r,\ r = \overline{1, N+1}$:

$$z_1(t)=\sum_{j=2}^{N}\left\{X_1(t)\int_{t_0}^{t}X_1^{-1}(\tau)\Phi(\tau)d\tau\left[[I-G(\Delta_N)]^{-1}V_j(\Delta_N)+\int_{t_{j-1}}^{t_j}\Psi(s)ds\right]\right\}\lambda_j+$$

$$+X_1(t)\int_{t_0}^{t}X_1^{-1}(\tau)[\Phi(\tau)[I-G(\Delta_N)]^{-1}V_1(\Delta_N)+B(\tau)]d\tau\,\lambda_1+$$

$$+X_1(t)\int_{t_0}^{t}X_1^{-1}(\tau)[\Phi(\tau)[I-G(\Delta_N)]^{-1}\mathrm{g}(\mathrm{F},\Delta_N)+F(\tau)]d\tau\,,t\in[t_0,t_1), \quad (26)$$

$$z_r(t)=\sum_{j=2}^{N}\left\{X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)\Phi(\tau)d\tau\left[[I-G(\Delta_N)]^{-1}V_j(\Delta_N)+\int_{t_{j-1}}^{t_j}\Psi(s)ds\right]\right\}\lambda_j+$$

$$+X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)A(\tau)d\tau\,\lambda_r+X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)[\Phi(\tau)[I-G(\Delta_N)]^{-1}V_1(\Delta_N)+B(\tau)]d\tau\,\lambda_1+$$

$$+X_r(t)\int_{t_{r-1}}^{t}X_r^{-1}(\tau)[\Phi(\tau)[I-G(\Delta_N)]^{-1}\mathrm{g}(\mathrm{F},\Delta_N)+F(\tau)]d\tau\,,t\in[t_{r-1},t_r),\ \ \mathrm{r}=\overline{2,N}. \quad (27)$$

We introduce the notations

$$D_{r,j}(\Delta_N)=X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)\Phi(\tau)d\tau\left[[I-G(\Delta_N)]^{-1}V_j(\Delta_N)+\int_{t_{j-1}}^{t_j}\Psi(s)ds\right],\ \ r\neq j,\qquad \mathrm{r,j}=\overline{2,N},$$

$$D_{r,r}(\Delta_N)=X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)\Phi(\tau)d\tau\left[[I-G(\Delta_N)]^{-1}V_r(\Delta_N)+\int_{t_{j-1}}^{t_j}\Psi(s)ds\right]$$

$$++X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)A(\tau)d\tau\,,\mathrm{r}=\overline{2,N},$$

$$D_{r,1}(\Delta_N)=X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)\Phi(\tau)d\tau\,[I-G(\Delta_N)]^{-1}V_1(\Delta_N)+X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)B(\tau)d\tau\,,\mathrm{r}=\overline{1,N},$$

$$F_r(\Delta_N)=X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)\Phi(\tau)[I-G(\Delta_N)]^{-1}\mathrm{g}(\mathrm{F},\Delta_N)d\tau+X_r(t_r)\int_{t_{r-1}}^{t_r}X_r^{-1}(\tau)F(\tau)d\tau\,,\mathrm{r}=\overline{1,N}.$$

Then from (26), (27) we obtain

$$\lim_{t\to t_r-0}z_r(t)=\sum_{j=1}^{N}D_{r,j}(\Delta_N)\,\lambda_{\mathrm{j}}+F_r(\Delta_N),\quad \mathrm{r}=\overline{1,N}. \quad (28)$$

Substituting the corresponding right-hand sides of (28) into the conditions (16) – (18), we obtain a system of linear algebraic equations with respect to the parameters$\lambda_r,\ r=\overline{1,N+1}$:

$$\left[I+D_{N,N}(\Delta_N)\right]\lambda_N+\sum_{j=1}^{N-1}D_{N,j}(\Delta_N)\,\lambda_{\mathrm{j}}=-F_N(\Delta_N), \quad (29)$$

$$\sum_{\substack{j=1\\ j\neq 2}}^{N}D_{1,j}(\Delta_N)\,\lambda_{\mathrm{j}}-\left[I-D_{1,2}(\Delta_N)\right]\lambda_2=-F_1(\Delta_N), \quad (30)$$

$$\left[I + D_{s,s}(\Delta_N)\right]\lambda_s - \left[I - D_{s,s+1}(\Delta_N)\right]\lambda_{s+1} +$$

$$+\sum_{\substack{j=1 \\ j\neq s, j\neq s+1}}^{N} D_{s,j}(\Delta_N)\,\lambda_j = -F_s(\Delta_N), \quad s = \overline{2, N-1}. \tag{31}$$

We denote the matrix corresponding to the left side of the system of equations (29) - (31) by $Q_*(\Delta_N)$and write the system in the form

$$Q_*(\Delta_N)\lambda = -F_*(\Delta_N), \;\; \lambda \in R^{(P-1)N}, \tag{32}$$

where $F_*(\Delta_N) = (F_N(\Delta_N), F_1(\Delta_N), \dots, F_{N-1}(\Delta_N)) \in R^{(P-1)N}$.

Cauchy problems for ordinary differential equations on subintervals

$$\frac{dx}{dt} = A(t)x + P(t), \;\; x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \;\; \mathrm{r} = \overline{1, N} \tag{33}$$

are a significant part of proposed algorithm. Here $P(t)$ is either $(n \times n)$ matrix, or n vector, both continuous on $[t_{r-1}, t_r]$, $\mathrm{r} = \overline{1, N}$. Consequently, solution to problem (33) is a square matrix or a vector of dimension n.

Denote by $a(P, t)$ the solution to the Cauchy problem (33). Obviously,

$$a(P, t) = X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau)\mathrm{P}(\tau)d\tau, \;\; t \in [t_{r-1}, t_r],$$

where $X_r(t)$ is a fundamental matrix of differential equation (33) on the r-th interval.

We offer the following numerical implementation of algorithm based on the Runge – Kutta method of 4th order and Simpson's method.

1. Suppose we have a partition $\Delta_N: 0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$. Divide each r-th interval $[t_{r-1}, t_r]$, $\mathrm{r} = \overline{1, N}$, into N_r parts with step $h_r = (t_r - t_{r-1})/N_r$. Assume on each interval $[t_{r-1}, t_r]$ the variable $\hat{t}$ takes its discrete values: $\hat{t} = t_{r-1}, \hat{t} = t_{r-1} + h_r, \dots, \hat{t} = t_{r-1} + (N_r - 1)h_r, \hat{t} = t_r$, and denote by $\{t_{r-1}, t_r\}$ the set of such points.
2. Using the Runge Kutta method of 4th order, we find the numerical solutions to Cauchy problems

$$\frac{dx}{dt} = A(t)x + \Phi(t), \;\; x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r],$$

and define the values of $(n \times n)$ matrices $a^{h_r}(\Phi, \hat{t})$on the set$\{t_{r-1}, t_r\}$, $\mathrm{r} = \overline{1, N}$.

3. Using the values of $(n \times n)$ matrices $\Psi(s)$and $a^{h_r}(\Phi, \hat{t})$on $\{t_{r-1}, t_r\}$, and Simpson's method, we calculate the$(n \times n)$matrices

$$\widehat{\Psi}_r^{h_r}(\Phi) = \int_{t_{r-1}}^{t_r} \Psi(\tau) a^{h_r}(\Phi, \tau) d\tau, \;\; \mathrm{r} = \overline{1, N}.$$

Summing up the matrices $\widehat{\Psi}_r^{h_r}(\Phi)$ over r, we find the $(n \times n)$ matrices $G^{\tilde{h}}(\Delta_N) = \sum_{j=1}^{N} \widehat{\Psi}_j^{h_j}(\Phi)$, where $\tilde{h} = (h_1, h_2, \dots, h_N) \in R^P$.

4. Solving the Cauchy problems

$$\frac{dx}{dt} = A(t)x + A(t), \;\; x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r],$$

$$\frac{dx}{dt} = A(t)x + B(t), \;\; x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r],$$

$$\frac{dx}{dt} = A(t)x + F(t), \;\; x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \qquad \mathrm{r} = \overline{1, N},$$

by using again the Runge–Kutta method of 4th order, we find the values of $(n \times n)$ matrices $a(\mathrm{A}, \hat{t}), a(B, \hat{t})$ and n vector $a(F, \hat{t})$ on $\{t_{r-1}, t_r\}$, $\mathrm{r} = \overline{1, N}$.

5. Applying Simpson's method on the set $\{t_{r-1}, t_r\}$, we evaluate the definite integrals

$$\widehat{\Psi}_r^{h_r} = \int_{t_{r-1}}^{t_r} \Psi(\tau) d\tau, \qquad \widehat{\Psi}_r^{h_r}(\mathrm{A}) = \int_{t_{r-1}}^{t_r} \Psi(\tau) a^{h_r}(\mathrm{A}, \tau) d\tau,$$

$$\widehat{\Psi}_r^{h_r}(\mathrm{B}) = \int_{t_{r-1}}^{t_r} \Psi(\tau) a^{h_r}(\mathrm{B},\tau) d\tau, \qquad \widehat{\Psi}_r^{h_r}(F) = \int_{t_{r-1}}^{t_r} \Psi(\tau) a^{h_r}(F,\tau) d\tau, \ \mathrm{r} = \overline{1,N}.$$

By the equalities

$$V_1^{\tilde{h}}(\Delta_N) = \sum_{j=1}^N \widehat{\Psi}_j^{h_j}(\mathrm{B}),$$

$$V_r^{\tilde{h}}(\Delta_N) = \widehat{\Psi}_r^{h_r}(\mathrm{A}) + \sum_{j=1}^N \widehat{\Psi}_j^{h_j}(\Phi) \cdot \widehat{\Psi}_r^{h_r}, \ \mathrm{r} = \overline{2,N},$$

$$g^{\tilde{h}}(F,\Delta_N) = \sum_{j=1}^N \widehat{\Psi}_j^{h_j}(\mathrm{F}).$$

we define the $(n \times n)$ matrices $V_r^{\tilde{h}}(\Delta_N)$ and n vectors $g^{\tilde{h}}(F,\Delta_N)$, $\mathrm{r} = \overline{1,N}$.

6. Construct the system of linear algebraic equations with respect to parameters

$Q_*^{\tilde{h}}(\Delta_N)\lambda = -F_*^{\tilde{h}}(\Delta_N), \ \lambda \in R^{(P-1)N}$,(34)

Solving the system (34), we find $\lambda^{\tilde{h}}$. As noted above, the elements of $\lambda^{\tilde{h}}$=$(\lambda_1^{\tilde{h}}, \lambda_2^{\tilde{h}}, \dots, \lambda_N^{\tilde{h}})$ are the values of approximate solution to problem (12)-(18)at the left-end points of subintervals.

7. To define the values of approximate solution at the remaining points of set $\{t_{r-1}, t_r\}$, we first find

$$\xi^{\tilde{h}} = \left[I - G^{\tilde{h}}(\Delta_N)\right]^{-1} \sum_{r=1}^N V_r^{\tilde{h}}(\Delta_N)\,\lambda_r^{\tilde{h}} + \left[I - G^{\tilde{h}}(\Delta_N)\right]^{-1} g^{\tilde{h}}(\mathrm{F},\Delta_N).$$

and then solve the Cauchy problems

$$\frac{du}{dt} = A(t)u + \Phi(t)\left(\xi^{\tilde{h}} + \sum_{j=2}^N \widehat{\Psi}_j^{h_j} \cdot \lambda_j^{\tilde{h}}\right) + B(t)\lambda_1^{\tilde{h}} + F(t),$$

$$u(t_0) = 0, \quad t \in [t_0, t_1],$$

$$\frac{du}{dt} = A(t)u + \Phi(t)\left(\xi^{\tilde{h}} + \sum_{j=2}^N \widehat{\Psi}_j^{h_j} \cdot \lambda_j^{\tilde{h}}\right) + B(t)\lambda_1^{\tilde{h}} + F(t),$$

$$u(t_{r-1}) = \lambda_r^{\tilde{h}}, \quad t \in [t_{r-1}, t_r], \ \mathrm{r} = \overline{2,N}.$$

And the solutions to Cauchy problems are found by the Runge–Kutta method of 4th order. Thus,the algorithm allows us to find the numerical solution to the problem (9)-(11).

So, we propose the numerically approximate method for solving of the original problem (1)-(4).

Example. We consider a linear boundary value problem with a parameter for an integro-differential equation of parabolic type

$$\frac{\partial u}{\partial t} = a(x,t)\frac{\partial^2 u}{\partial x^2} + c(x,t)u + b(x,t)\mu(x) +$$

$$+\varphi(x,t)\int_0^T \psi(x,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega = (0,\omega) \times (0,T), \tag{35}$$

$$u(x,0) = 0, \ x \in [0,\omega], \tag{36}$$

$$u(x,T) = 0, \ x \in [0,\omega], \tag{37}$$

$$u(0,t) = \tilde{\psi}_1(t), \ u(\omega,t) = \tilde{\psi}_2(t), \ t \in [0,T], \tag{38}$$

where $\omega = 0.5, T = 0.1, a(x,t) = 1, c(x,t) = 0, b(x,t) = t^2 + 1, \varphi(x,t) = x^2, \psi(x,s) = s, \ f(x,t) = xe^{xt}\sin(10\pi t) + 10\pi e^{xt}\cos(10\pi t) - t^2 e^{xt}\sin(10\pi t) - (t^2+1)(x^3+1) - \frac{[e^{0.1}(\pi x^2 - 20\pi x + 100\pi^3) - 20\pi x]x^2}{(x^2+100\pi^2)^2}$, $\tilde{\psi}_1(t) = \sin(10\pi t), \tilde{\psi}_2(t) = \mathrm{e}^{0.5\mathrm{t}}\sin(10\pi t)$.

We take $h = 0.1$ and produce a discretization by x: $x_i = ih, i = \overline{0,5}$.

We introduce the notations$u_i(t) = u(ih,t), \mu_i = \mu(ih), f_i(t) = f(ih,t), i = \overline{0,5}$.

Problem (35) - (38) is replaced by the following linear boundary value problem with a parameter for an integro-differential equation

$$\frac{du_i}{dt} = \frac{u_{i+1}-2u_i+u_{i-1}}{0.01} + (t^2+1)\mu_i + (0.1\cdot i)^2 \int_0^{0.1} s\cdot u_i(s)ds + f_i(t),\ \overline{1,4}, \tag{39}$$

$$u_i(0) = 0,\ i = \overline{0,5}, \tag{40}$$

$$u_i(0.1) = 0,\ i = \overline{0,5}, \tag{41}$$

$$u_0(t) = \sin(10\pi t),\ \ u_5(t) = e^{0.5t}\sin(10\pi t),\ \ t \in [0,0.1]. \tag{42}$$

In view of condition (40)-(42), from (39) we obtain two groups of equations for determining $\mu_0$andμ_5:

$b_0(0)\mu_0 = \dot{\tilde{\psi}}_1(0) - \varphi_0(0)\int_0^T \psi_0(s)\tilde{\psi}_1(s)ds - f_0(0)$,then$\mu_0 = 1$,

$b_5(0)\mu_5 = \dot{\tilde{\psi}}_2(0) - \varphi_5(0)\int_0^T \psi_5(s)\tilde{\psi}_2(s)ds - f_5(0)$,then$\mu_5 = 1.125$.

The functions$u_0(t),\ u_5(t)$, and parameters μ_0,μ_5 are known.

Problem (39)-(42) will be rewritten in vector-matrix form

$$\frac{du}{dt} = A(t)u + B(t)\mu + \Phi(t)\int_0^{0.1}\Psi(s)u(s)ds + F(t), \quad u,\mu \in R^4, \quad t \in (0,0.1), \tag{43}$$

$$u(0) = 0, \tag{44}$$

$$u(0.1) = 0, \tag{45}$$

where $u(t) = \big(u_1(t), u_2(t), u_3(t), u_4(t)\big)$, $\mu = (\mu_1,\mu_2,\mu_3,\mu_4)$ -unknown function and parameter,

$$A(t) = \begin{pmatrix} -200 & 100 & 0 & 0 \\ 100 & -200 & 100 & 0 \\ 0 & 100 & -200 & 100 \\ 0 & 0 & 100 & -200 \end{pmatrix},\ B(t) = \begin{pmatrix} t^2+1 & 0 & 0 & 0 \\ 0 & t^2+1 & 0 & 0 \\ 0 & 0 & t^2+1 & 0 \\ 0 & 0 & 0 & t^2+10 \end{pmatrix},$$

$$\Phi(t) = \begin{pmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.04 & 0 & 0 \\ 0 & 0 & 0.09 & 0 \\ 0 & 0 & 0 & 0.16 \end{pmatrix}, \Psi(s) = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix},$$

$$F(t) = \begin{pmatrix} f(0.1,t) + 100\sin(10\pi t) \\ f(0.2,t) \\ f(0.3,t) \\ f(0.4,t) + 100e^{0.5t}\sin(10\pi t) \end{pmatrix}.$$

Here we use thenumerical implementation of algorithm. Accuracy of solution depends on the accuracy of solving the Cauchy problem on subintervals andevaluating definite integrals. We provide the results of the numerical implementation of algorithm by partitioning theinterval [0, 0.1] with step h = 0.002.

Solution to problem (35)-(38) is pair $(u^*(x,t), \mu^*(x))$, where $u^*(x,t) = e^{xt}\sin(10\pi t)$, $\mu^*(x) = x^3 + 1$. Then solution to problem (43)-(45) is pair $(u^*(t), \mu^*)$, where $u^*(t) = \begin{pmatrix} e^{0.1t}\sin(10\pi t) \\ e^{0.2t}\sin(10\pi t) \\ e^{0.3t}\sin(10\pi t) \\ e^{0.4t}\sin(10\pi t) \end{pmatrix}$, $\mu^* = \begin{pmatrix} 1.001 \\ 1.008 \\ 1.027 \\ 1.064 \end{pmatrix}$ and the following estimates $max\|\mu^* - \tilde{\mu}\| < 0.00009$, and $\max_{j=\overline{0,50}}\|u^*(t_j) - \tilde{u}(t_j)\| < 0.000004$ is true.

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ПАРАБОЛАЛЫҚ ТЕКТЕС ИНТЕГРАЛДЫҚ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ҮШІН БАСҚАРУ ЕСЕБІН ШЕШУДІҢ САНДЫҚ ЖУЫҚТАЛҒАН ӘДІСІ

Аннотация. Параболалық тектес интегралдық-дифференциалдық теңдеулер үшін параметрі бар сызықтық шеттік есеп зерттеледі. Кеңістіктік айнымалыны дискреттеу көмегімен қарастырылатын есеп жәй интегралдық-дифференциалдық теңдеулер жүйесі үшін параметрі бар сызықтық екі нүктелі шеттік есеппен аппроксимацияланады. Алынған есепті шешу үшін параметрлеу әдісі қолданылады. Аппроксимацияланған есеп Фредгольм интегралдық дифференциалдық теңдеулер жүйесі үшін арнайы Коши есептерінен, шеттік шарттардан және бөлу нүктелерінде шешімнің үзіліссіз шарттарынан тұратын пара-пересепке келтіріледі. Параметрлері бар жәй дифференциалдық теңдеулер жүйесі үшін Коши есебін шешу дифференциалдық теңдеудің фундаменталдық матрицасы көмегімен құрылады. Параметрлерге қатысты сызықты алгебралық теңдеулер жүйесі тиісті нүктелердің мәндерін шеттік шарт пен үзіліссіз шарттарына қою арқылы құрылады.Есепті шешудің құрылған жүйе мен ішкі аралықтардаКоши есебін шешудің4-ші ретті Рунге-Куттаәдісіне негізделген сандық әдісі ұсынылады.

Кілттік сөздер: параболикалық тектес дербес туындылы интегралдық-дифференциалдық теңдеулер, параметрі бар есеп, аппроксимация, сандық жуықталған әдіс, алгоритм.

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ЧИСЛЕННО ПРИБЛИЖЕННЫЙ МЕТОД РЕШЕНИЯ ЗАДАЧИ УПРАВЛЕНИЯ ДЛЯ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ПАРАБОЛИЧЕСКОГО ТИПА

Аннотация. Исследуется линейная краевая задача с параметром для интегро-дифференциальных уравнений параболического типа. С помощью дискретизации пространственной переменной рассматриваемая задача аппроксимируется линейной двухточечной краевой задачей с параметром для системы интегро-дифференциальных уравнений. Для решения полученной задачи применяется метод параметризации. Аппроксимирующая задача сведена к эквивалентной задаче, состоящей из специальной задачи Коши для системы интегро-дифференциальных уравнений Фредгольма, краевых условий и условий непрерывности решения в точках разбиения.Решение задачи Коши для системы обыкновенных дифференциальных уравнений с параметрами строится с использованием фундаментальной матрицы дифференциального уравнения. Система линейных алгебраических уравнений относительно параметров составляется путем подстановки значений соответствующих точек в краевое условие и условия непрерывности. Предлагается численный метод решения задачи, основанный на решении построенной системы и метода Рунге-Кутты 4-го порядка для решения задачи Коши на подинтервалах.

Ключевые слова: интегро-дифференциальные уравнения с частными производными параболического типа, задача с параметром, аппроксимация, численно приближенный метод, алгоритм.

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REFERENCES

[1] Sloan I. and Thomée V. Time discretization of an integro-differential equation of parabolic type // SIAM Journal on Numerical Analysis, 23(1986), 1052–1061.

[2] Cannon J. and LinY. Non-classical H1 projection and Galerkin methods for non-linear parabolic integro-differential equations // Calcolo, 25(1988), 187–201.

[3] Thomée V. and ZhangN.-Y. Error estimates for semidiscrete finite element methods for parabolic integro-differential equations // Mathematics of Computation, 53(1989), 121–139.

[4] LinY., ThoméeV., and WahlbinL. Ritz-Volterra projections to finite element spaces and applications to integro-differential and related equations // SIAM Journal on Numerical Analysis, 28(1991), 1047–1070.

[5] Pani A. and PetersonE. Finite element methods with numerical quadrature for parabolic integrodifferential equations // SIAM Journal on Numerical Analysis, 33(1996), 1084–1105.

[6] LinY. Semi-discrete finite element approximations for linear parabolic integrodifferential equations with integrable kernels // Journal of Integral Equations and Applications, 10(1998), 51–83.

[7] Pani A. and SinhaR. Finite element approximation with quadrature to a time dependent parabolic integro-differential equation with nonsmooth initial data // Journal of Integral Equations and Applications, 13(2001), 35–72.

[8] Pani A. and FairweatherG. H1-Galerkin mixed finite element methods for parabolic partial integro-differential equations // IMA Journal of Numerical Analysis, 22(2002), 231–252.

[9] McLean W., Sloan I. and Thomée V. Time discretization via Laplace transformation of an integro-differential equation of parabolic type // IMA Journal of Numerical Analysis, 24(2004), 439–463.

[10] Matache A.-M., Schwab C. and Wihler T. Fast numerical solution of parabolic integrodiffe-rential equations with applications in finance // SIAM Journal on Scientific Computing, 27(2005), 369–393.

[11] Sinha R., Ewing R. and LazarovR. Some new error estimates of a semidiscrete finite volume element method for a parabolic integro-differential equation with nonsmooth initial data // SIAM Journal on Numerical Analysis, 43(2006), 2320–2344.

[12] Zhang N. On fully discrete Galerkin approximations for partial integrodifferential equations of parabolic type // Mathematics of Computation, 60(1993), 133–166.

[13] Dolejší V. and Vlasák M. Analysis of a BDF-DGFE scheme for nonlinear convection-diffu-sion problems // NumerischeMathematik, 110(2008), 405–447.

[14] Volpert V. Elliptic partial differential equations Vol. 2: Reaction-Diffusing Equations, Birkhauser Springer, Basel etc., 2014.

[15] Douglas J. and Jones B. Numerical methods for integro-differential equations of parabolic and hyperbolic types // NumerischeMathematik, 4(1962), 96–102.

[16] Yanik E.and FairweatherG. Finite element methods for parabolic and hyperbolic partial integro-differential equations // Nonlinear Analysis, 12(1988), 785–809.

[17] Pani A., ThoméeV., and Wahlbin L. Numerical methods for hyperbolic and parabolic integro-differential equations// Journal of Integral Equations and Applications, 4(1992), 533–584.

[18] Dzhumabayev D.S. Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation// U.S.S.R. Computational Mathematics and Mathematical Physics. 1989. Vol. 29, No. 1. P. 34-46.

[19] Dzhumabaev D.S. A method for solving the linear boundary value problem for an integro-differential equation // Computational Mathematics and Mathematical Physics. 2010. Vol. 50, No. 7. P. 1150-1161.

[20] Dzhumabaev D.S. An algorithm for solving a linear two-point boundary value problem for an integro-differential equation // Computational Mathematics and Mathematical Physics. 2013. Vol. 53, No. 6. P. 736-758.

[21] Dzhumabaev, D.S., Bakirova, E.A. Criteria for the unique solvability of a linear two-point boundary value problem for systems of integro-differential equations // Differential Equations. 2013. Vol. 49, No. 9. P. 1087-1102.

[22] Dzhumabaev D.S. Necessary and sufficient conditions for the solvability of linear boundary-value problems for the Fredholmintegro-differential equations // Ukranian Mathematical Journal. 2015. Vol. 66, No. 8. P. 1200-1219.

[23] Dzhumabaev D.S. Solvability of a linear boundary value problem for a Fredholmintegro-differential equation with impulsive inputs // Differential Equations. 2015. Vol. 51, No. 9. P. 1180-1196.

[24] Dzhumabaev D.S. On one approach to solve the linear boundary value problems for Fredholmintegro-differential equations // Journal of Computational and Applied Mathematics. 2016. Vol. 294, P. 342-357.

[25] Dzhumabaev D.S. Bakirova, E.A., KadirbayevaZh.M. An algorithm for solving a control problem for a differential equation with a parameter // News of the NAS RK. Phys.-Math. Series. 2018. Vol. 5, No. 321. P.25-32. DOI: https://doi.org/10.32014/2018.2518-1726.4

[26] Assanova A.T., Bakirova E.A., KadirbayevaZh.M. Numerical implementation of solving a boundary value problem for a system of loaded differential equations with parameter // News of the NAS RK. Phys.-Math. Series. 2019. Vol. 3, No. 325. -P. 77-84. DOI: https://doi.org/10.32014/2019.2518-1726.27

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INVESTIGATION OF DIFFERENT DARK MATTER PROFILES

Abstract. Different, commonly accepted and widely used phenomenological dark matter profiles such as the pseudo-isothermal sphere, Burkert, Navarro-Frenk-White, Moore and Einasto profiles are employed to estimate the mass distribution of dark matter in various galaxies. The Newtonian gravity is involved to perform computations at large galactic scales. The distribution of dark matter in diverse types of galaxies is assumed to be spherical without taking into account the complex structure of galaxies such as their cores, inner and outer bulges, disks and halos. The theoretical rotation curves are overlapped with the observations for each individual galaxy. By means of the least square algorithm the model parameters are inferred from the observational data and are subjected to the Bayesian information criterion, which identifies the more preferred model. The masses of dark matter are calculated for each galaxy, with all listed profiles, and compared with the visible masses of the galaxies. The results are in agreement with the ones in the literature.
Keywords: galaxies, rotation curves, dark matter.

Introduction

Astronomers face the fundamental problem of a mass deficiency, which is known as the dark matter (DM) problem at galactic scales. One observes the presence of DM from dynamical astronomical measurements, but up to now no one was able to detect any DM particle yet in the ground based laboratories. The only knowledge we possess is the fact that DM interacts with an ordinary matter via gravity and dominates preferentially on large scales. It is therefore of particular importance to understand its properties to set guidelines in preparing more focused physics experiments to detect it. The determination of the mass distribution in galaxies is one of the most basic subjects in galactic astronomy, and is usually obtained by analyzing rotation curves.

The idea of DM does not alter the law of gravity like Modified Newtonian Dynamics [1], but proposes that there exists a new type of matter that has yet to be identified. This idea is also supported by gravitational lensing experiments [2]. DM does not participate in electromagnetic and strong interactions; otherwise it would have already been detected. It is believed that DM can interact weakly if it is composed of weakly interacting particles [3], though there is no evidence to support this idea. So far, observations indicate that DM acts through gravitational interactions and seems to behave frictionless. Due to this frictionless behavior, DM is predicted by N-body simulations to be distributed in spherically symmetric halos [4].

To investigate the distribution of DM within any galaxy a number of different phenomenological density profiles have been proposed in the literature [5]:

- Cored profiles with central density ρ_0 and scale radius r_0 with $x = r/r_0$ being the dimensionless radial distance (coordinate) from the center of a galaxy to the considered point:

– pseudo-isothermal (ISO) sphere profile:

$$\rho_{Iso}(r) = \frac{\rho_0}{1+x^2} \ , \tag{1}$$

– Burkert profile:

$$\rho_{Bur}(r) = \frac{\rho_0}{(1+x)(1+x^2)} \ . \tag{2}$$

- Cusped profiles with characteristic radius r_0 where the density profile has a logarithmic slope of −2 (the "isothermal" value) and ρ_0 as the local density at that radius.

– Navarro–Frenk–White profile:

$$\rho_{NFW}(r) = \frac{\rho_0}{x(1+x)^2} \tag{3}$$

– Moore profile:

$$\rho_{Moo}(r) = \rho_0 x^{-1.16}(1+x)^{-1.84} \tag{4}$$

- – Einasto profile with an extra parameter *n* which determines the degree of curvature (shape) of the profile. The family of Einasto profiles with relatively large indices *n*>4 are identified with cuspy halos, while low index values *n*<4 presents a cored-like behavior. The lower the index *n*, the more cored-like the halo profile:

$$\rho_{Ein}(r) = \rho_0 \exp\left\{2n\left(1 - x^{\frac{1}{n}}\right)\right\} \ . \tag{5}$$

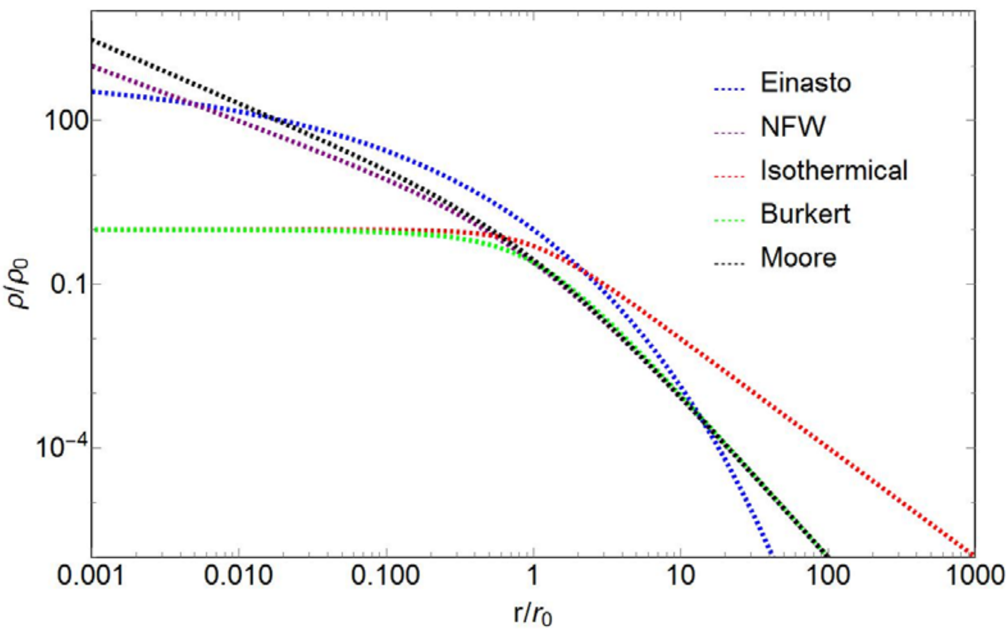

Figure 1 - Different phenomenological dark matter density profiles

In the case of Milky Way Galaxy these profiles have two constraints. They are anchored at a distance of 8.33 kilo parsec (kpc), which is the distance from the center of Milky Way Galaxy to the Sun. The second constraint is the total predicted mass of the distribution inside a radius of 60 kpc. This mass has been predicted to be $M = 4.7 \cdot 10^{11} M_{\odot}$. The comparison among the profiles is shown in Figure 1. The profiles follow a similar shape when the distance from the galactic center is larger than that of the Sun and differ more towards the center of the galaxy. The Burkert and isothermal profiles become constant and approach their characteristic density, while the NFW and Moore profiles diverge near the center.

Methods

The problem of the DM distribution in galaxies, as usually addressed in the literature, is mainly focused in the halo regions and associated with the galaxy rotation curves obtained from the observations [6]. A galactic rotation curve describes how the rotation velocity of objects in the galaxy changes as a function of the object's distance to the center. As an example we consider a central mass with a test particle moving on a circular path in the field of the former. The rotational velocity can be determined in this example from Newton's gravitational equations and the centrifugal force:

$$F = G\frac{M(r)\cdot m}{r^2} = \frac{m\cdot v^2(r)}{r} \quad \Rightarrow \quad v(r) = \sqrt{\frac{GM(r)}{r}} \tag{6}$$

where $M(r)$ is the central mass profile, m is the test mass and r is the distance between the central mass and the test mass. The rotational velocity is represented by $v(r)$.

The mass of the galaxy is not distributed in a central point, so one has to consider the actual distribution. The calculation for the galactic distribution requires the integration of Newton's gravitational equations with the entire distribution taken into account. We adopt a circular motion of a test mass around the center of the galaxy. The expected rotation curve would then increase quickly since the considered mass increases rapidly, due to the high density near the center. Based on the visible mass, the rotation curve is expected to eventually drop off near the edge of the galaxy like $1/\sqrt{r}$ as in formula (6). This is however not what is measured in astronomy. The rotation curves do not drop off according to Newton's laws but stay flat near the edge of the galaxies [7]

In equation (6) we calculate the mass profile $M(r)$ with the help of the following formula:

$$M(r) = \int_0^r \rho(r')\cdot 4\pi\, r'^2 dr' \qquad (7),$$

where $\rho(r)$ is the DM density profile taken from Eqs. (1) through (5).

For convenience, we choose such units where the mass is given in the units of one solar mass and the radial distance is in parsecs:

$$v(r) = 10^{-5}\sqrt{\frac{G\cdot M(r)\cdot M_{sun}}{r\cdot pc}} \tag{8}$$

where $M_{sun} = M_{\odot}$ is the mass of the Sun in grams, *pc* is parsec in cm, and the numerical factor in front of the square root converts the units of $v(r)$ into km/s.

In order to carry out the fitting procedure we exploit the Levenberg-Marquardt nonlinear least squares method [8, 9].

Results

We compare the fits of rotation curves obtained by means of the models considered in the previous section. Since one has to compare the models with different number of parameters, which are not nested into each other, we use the Bayesian Information Criterion (BIC) formulated and developed by Schwarz [10]. It provides a penalty to models with larger number of parameters to check which of them is more likely to be realistic. A model with a minimum BIC value is favored [5]. The results of the fitting procedure are presented in Tables 1 - 6 and in Figs. 2 - 7.

Table 1 - Model parameters for irregular dwarf galaxy DDO 154

The mass of the baryonic (luminous) matter in the galaxy is $3.58 \cdot 10^8$ $M_\odot$ [11]					
Profiles	$\rho_0 \pm \delta\rho_0$, $10^{-3}\frac{M_\odot}{pc^3}$	$r_0 \pm \delta r_0$, kpc	$M \pm \delta M$, $M_\odot$	n	BIC
Burkert	32.74 ± 1.47	2.44 ± 0.07	$(7.59 \pm 0.61) \cdot 10^8$	-	149
NFW	1.61 ± 0.35	14.46 ± 2.41	$(1.18 \pm 0.49) \cdot 10^{10}$	-	207
ISO	34.74 ± 1.89	1.31 ± 0.05	$(2.07 \pm 0.22) \cdot 10^8$	-	141
Moore	0.31 ± 0.18	38.96 ± 18.21	$(5.23 \pm 6.14) \cdot 10^{10}$	-	235
Einasto	2.73 ± 0.27	4.53 ± 0.25	$(2.14 \pm 0.33) \cdot 10^9$	1.72 ± 0.13	119

As one can see from table 1, for DDO 154 galaxy the BIC value is minimum for the Einasto profile and is maximum for the Moore profile. The mass obtained by means of the Einasto profile is one order of magnitude larger than the visible mass.

Table 2 - Model parameters for spiral galaxy NGC 1560

The mass of the baryonic matter in the galaxy is $8.2 \cdot 10^8$ $M_\odot$ [12]					
Profiles	$\rho_0 \pm \delta\rho_0$, $10^{-3}\frac{M_\odot}{pc^3}$	$r_0 \pm \delta r_0$, kpc	$M \pm \delta M$, $M_\odot$	n	BIC
Burkert	47.81 ± 2.71	3.12 ± 0.12	$(2.32 \pm 0.24) \cdot 10^9$	-	142
NFW	2.61 ± 0.47	17.32 ± 2.38	$(3.31 \pm 1.14) \cdot 10^{10}$	-	136
ISO	47.11 ± 2.93	1.76 ± 0.08	$(6.98 \pm 0.85) \cdot 10^8$	-	130
Moore	0.54 ± 0.22	43.81 ± 13.96	$(1.32 \pm 1.06) \cdot 10^{11}$	-	140
Einasto	1.75 ± 1.28	9.36 ± 4.09	$(1.32 \pm 1.57) \cdot 10^{10}$	2.86 ± 1.00	135

For NGC 1560 galaxy the BIC value is minimum for the isothermal profile and is maximum for the Burkert profile, though the difference between the values is not large. It is worth noting that the modest improvement in the BIC of the isothermal sphere model with respect to the Einasto and NFW is not that strong to assess that the isothermal profile has to be preferred.

Table 3 - Model parameters for intermediate spiral galaxy NGC 2403

The mass of the baryonic matter in the galaxy is $2.58 \cdot 10^9$ $M_\odot$ [11]					
Profiles	$\rho_0 \pm \delta\rho_0$, $10^{-3}\frac{M_\odot}{pc^3}$	$r_0 \pm \delta r_0$, kpc	$M \pm \delta M$, $M_\odot$	n	BIC
Burkert	207.94 ± 10.16	2.77 ± 0.07	$(7.06 \pm 0.51) \cdot 10^9$	-	1252
NFW	32.64 ± 1.11	6.91 ± 0.13	$(2.62 \pm 0.14) \cdot 10^{10}$	-	764
ISO	521.34 ± 22.83	0.84 ± 0.02	$(8.27 \pm 0.6) \cdot 10^8$	-	842
Moore	15.66 ± 0.57	9.41 ± 0.18	$(3.79 \pm 0.21) \cdot 10^{10}$	-	732
Einasto	4.35 ± 0.43	9.43 ± 0.48	$(3.76 \pm 0.56) \cdot 10^{10}$	5.98 ± 0.27	692

For NGC 2403 galaxy the BIC value is minimum for the Einasto profile and is maximum for the Burkert profile. The mass obtained by the Einasto profile is one order of magnitude larger than the visible mass.

Table 4 - Model parameters for unbarred spiral galaxy NGC 2976

The mass of the baryonic matter in the galaxy is $1.36 \cdot 10^8$ $M_\odot$ [11]					
Profiles	$\rho_0 \pm \delta\rho_0$, $10^{-3} \frac{M_\odot}{pc^3}$	$r_0 \pm \delta r_0$, kpc	$M \pm \delta M$, $M_\odot$	n	BIC
Burkert	273.17 ± 12.6	1.7 ± 0.08	$(2.14 \pm 0.24) \cdot 10^9$	-	126
NFW	0.77 ± 3.73	138.45 ± 664.57	$(0.49 \pm 5.61) \cdot 10^{13}$	-	174
ISO	238.66 ± 11.52	1.07 ± 0.05	$(7.91 \pm 1.04) \cdot 10^8$	-	129
Moore	0.05 ± 2.1	758.54 ± 3040.75	$(0.06 \pm 2.72) \cdot 10^{15}$	-	202
Einasto	35.5 ± 12.3	2.52 ± 0.61	$(4.39 \pm 2.72) \cdot 10^9$	1.1 ± 0.3	128

For NGC 2976 galaxy the BIC value is minimum for the Burkert profile and is maximum for the Moore profile. As before the small difference in the BIC with respect to the Einasto, isothermal and Burkert profiles does not allow to assess which model is better (they are comparable with each other). For this galaxy the mass inferred by the Burkert profile is ten times larger than the visible mass.

Table 5 - Model parameters for spiral galaxy NGC 3627

The mass of the baryonic matter in the galaxy is $8.18 \cdot 10^8$ $M_\odot$ [11]					
Profiles	$\rho_0 \pm \delta\rho_0$, $10^{-3} \frac{M_\odot}{pc^3}$	$r_0 \pm \delta r_0$, kpc	$M \pm \delta M$, $M_\odot$	n	BIC
Burkert	2660 ± 477	1.16 ± 0.09	$(6.61 \pm 0.24) \cdot 10^9$	-	53
NFW	1670 ± 498	1.47 ± 0.21	$(1.28 \pm 0.53) \cdot 10^{10}$	-	60
ISO	43938 ± 148672	0.13 ± 0.22	$(2.57 \pm 7.79) \cdot 10^{10}$	-	86
Moore	1087 ± 343	1.71 ± 0.24	$(1.57 \pm 0.69) \cdot 10^{10}$	-	61
Einasto	147.4 ± 9.9	2.71 ± 0.11	$(2.21 \pm 0.25) \cdot 10^{10}$	1.04 ± 0.13	30

For NGC 3627 galaxy the BIC value is minimum for the Einasto profile and is maximum for the isothermal profile. Here the mass inferred by the Einasto profile is one order of magnitude larger than the visible mass.

Table 6 - Model parameters for spiral galaxy NGC 5585

The mass of the baryonic matter in the galaxy is $1.6 \cdot 10^9$ $M_\odot$ [13]					
Profiles	$\rho_0 \pm \delta\rho_0$, $10^{-3} \frac{M_\odot}{pc^3}$	$r_0 \pm \delta r_0$, kpc	$M \pm \delta M$, $M_\odot$	n	BIC
Burkert	89.6 ± 2.3	2.83 ± 0.05	$(3.24 \pm 0.14) \cdot 10^9$	-	38
NFW	6.02 ± 1.4	12.56 ± 2.02	$(2.89 \pm 1.2) \cdot 10^{10}$	-	76
ISO	90.5 ± 4.5	1.52 ± 0.05	$(8.57 \pm 0.82) \cdot 10^8$	-	51
Moore	1.79 ± 0.77	23.69 ± 7.27	$(6.93 \pm 4.85) \cdot 10^{10}$	-	81
Einasto	10.26 ± 0.61	4.48 ± 0.12	$(7.3 \pm 0.6) \cdot 10^9$	1.27 ± 0.08	33

For NGC 5585 galaxy the BIC value is minimum for the Einasto profile and is maximum for the Moore profile. Here as before the BIC is similar between the Einasto and Burkert models. The mass inferred by the Einasto profile has the same order as the visible mass.

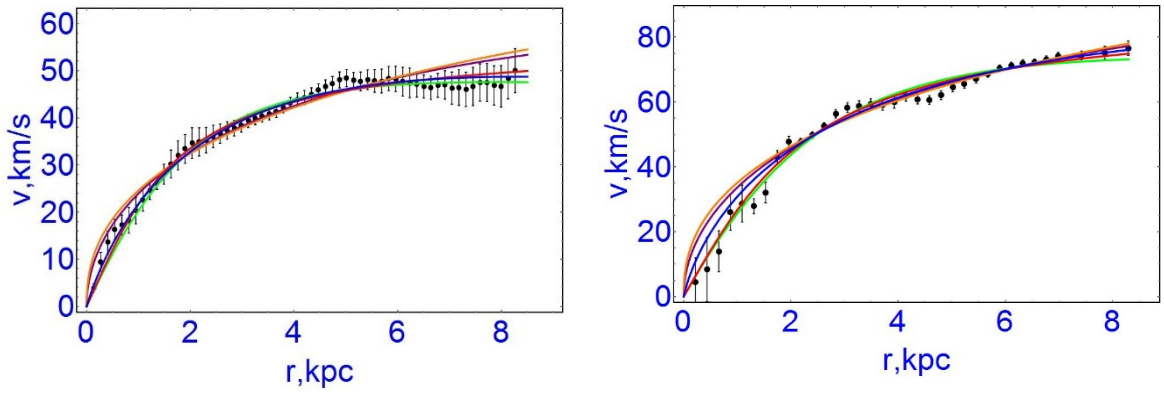

Figure 2 - Rotation curves of galaxies and fitted models.
Left panel: galaxy DDO 154. Right panel: galaxy NGC 1560

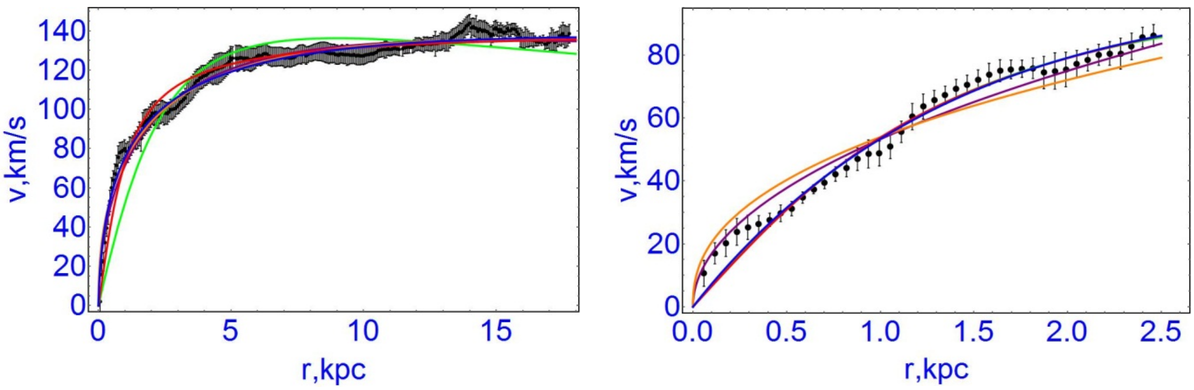

Figure 3 - Rotation curves of galaxies and fitted models.
Left panel: galaxy NGC2403. Right panel: galaxy NGC 2976

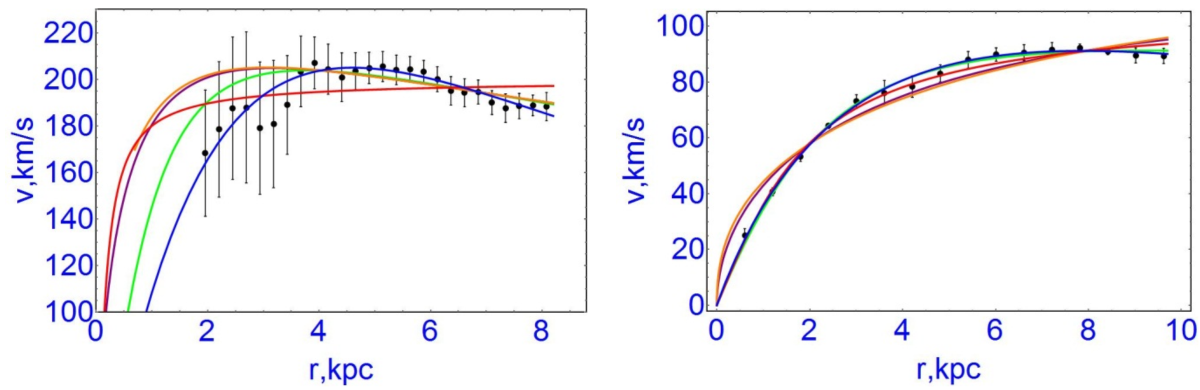

Figure 4 - Rotation curves of galaxies and fitted models.
Left panel: galaxy NGC 3627. Right panel: galaxy NGC 5585.

In figures 2-4 the black thick points show observational data with their error bars for one dwarf and five spiral galaxies, the green curves show the Burkert profile, the purple curves show the NFW profile, the red curves show the isothermal profile, the orange curves show the Moore profile and the blue curves show the Einasto profile (color online). The theoretical curves were obtained by the fitting procedure. Using the method of least squares we calculated the numerical values of the scale radius and characteristic density. In the case of the Einasto profile, the value of the Einasto index was also obtained. In the end we calculated the mass of DM in the galaxies.

Discussion and Conclusion

We have analyzed one dwarf galaxy and five spiral galaxies in this work. The complex structure of galaxies was abandoned and the DM distribution was adopted to be spherical for a given galaxy. The rotation curves constructed in this work have been compared with the analogous results in the literature for galaxies NGC 2403, NGC 3627, NGC 2976, DDO 154 in Ref. [11], and for galaxies NGC 1560 and NGC 5585 in Refs. [12] and [13], respectively.

The model parameters were inferred from the observational data points by using "NonLinearModelFit" command in the scientific software "Wolfram Mathematica 11". The command allows one to automatically conduct the fitting procedure for a given model and data points. As a result, the total mass of DM was estimated in the considered galaxies and confronted with their luminous masses. The DM mass, in many cases, is larger than the visible mass, as expected.

The Einasto model showed the best results in four cases (DDO154, NGC 2403, NGC 3627, NGC 5585), Burkert and Isothermical profiles showed good results for NGC 2976 and NGC 1560 galaxies, respectively, as the BIC value was the lowest compared to other models. Unlike other models, the Einasto model depends upon three parameters and one gets a better fit. Therefore, there is a higher compliance with the observations. The NFW profile showed small deviations from the observed galaxies NGC 5585, NGC 1560, NGC 2976. In other profiles we have no noticeable differences.

Overall, all models considered in this work showed good results, though in some cases not all the data points were covered with the theoretical curves. These discrepancies can be caused by the fact that we simply ignored the complex structure, the baryonic constituents in galaxies. If we took into account the inner structure of galaxies as in Ref. [14] and constituents of the matter as in Ref [15], probably the fits would be much better than the ones presented in this work.

Nevertheless there are still a plenty of open issues related to DM. Although one can observe indirectly the presence of DM in any galaxy or clusters, we still do not know why the DM distribution is so different from one galaxy to another. For example, the recent discovery of DM distribution in the Markarian galaxy shows that DM is mainly concentrated in the core of the galaxy [16]. Another example shows a galaxy with barely no DM [17].

The nature of DM particles still remains mystery, though there are some theoretical models predicting the whole set of DM particles, see Refs. [3, 18] for details. The role of DM in the evolution of the universe is yet to be understood.

It will be interesting to investigate other galaxies [19], taking into account their intricate structure, and involving latest theoretical models. This issue will be addressed in future studies.

Acknowledgement

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APPENDIX

The rotation curve data points for galaxies NGC 2403, NGC 3627, NGC 2976, DDO 154 are given in Ref. [11], and for galaxies NGC 1560, NGC 5585 are taken from [12] and [13], respectively.

In the observational data, the distance from the center to the point R is represented in angular seconds. 1 angular second is equal to $\frac{1}{3600}$ part of a degree. To convert an angular second into a kiloparsec, one should use the following general formula:

$$R = a \cdot tg(\varphi) \tag{9}$$

where φ is the specified distance in angular seconds, a is the distance between our galaxy (Earth) and the galaxy under consideration. We express φ in radians. Finally for R we obtain:

$$R=a\cdot tg\left(\frac{1}{3600}\cdot\frac{\pi}{180^{\circ}}\varphi\right) \tag{10}$$

Table 7 - The distance between Milky Way Galaxy (Earth) and the galaxy under consideration

Galaxies	a , [kpc]	References
DDO 154	4300	[11]
NGC 1560	3000	[12]
NGC 5585	6200	[13]
NGC 2403	3180	[20]
NGC 3627	10100	[21]
NGC 2976	3450	[22]

ӘОЖ 524.7, 524-1:629.78
ХҒТАР 41.27.29, 89.51.17

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ҚАРАҢҒЫ МАТЕРИЯНЫҢ ӘР ТҮРЛІ ПРОФИЛЬДЕРІН ЗЕРТТЕУ

Аннотация. Әр түрлі галактикалардағы қараңғы материяның массасы таралуын бағалау үшін әдебиетте белгілі, кеңінен пайдаланылатын Изотермал, Буркерт, Наварро-Френк-Уайт, Мур және Эйнасто профильдері қолданылады. Ньютонның гравитация теориясы ауқымды галактикалық масштабтардағы есептеулерді орындау үшін жұмылдырылды. Алуан түрлі галактикаларда қараңғы материяның таралуы (үлестірілуі) сфералық деп ұйғарылды және галактикалардың күрделі құрылымы, яғни олардың ядросы, ішкі және сыртқы балджадары, дискілері мен галолары ескерілмеді. Теориялық айналу қисықтары галактикалар үшін бақылау мәліметтерімен сәйкестендірілді. Ең кіші квадраттар әдісі көмегімен үлгілердің параметрлері бақылау деректерінен есептеліп алынды және неғұрлым сәйкес келетін үлгіні анықтау үшін Байестік Информациялық Критерий қолданылды. Қараңғы материяның массасы барлық көрсетілген профильдер арқылы әр дербес галактика үшін есептелінді және галактикалардың көрінетін массасымен салыстырылады. Нәтижелер әдебиеттегі мәліметтерімен сәйкес келеді.

Түйін сөздер: галактикалар, айналу қисықтары, қараңғы материя.

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ИССЛЕДОВАНИЕ РАЗЛИЧНЫХ ПРОФИЛЕЙ ТЕМНОЙ МАТЕРИИ

Аннотация. Различные, общепринятые и широко используемые феноменологические профили темной материи, такие как псевдоизотермическая сфера, профили Буркерта, Наварро-Френка-Уайта, Мура и Эйнасто, используются для оценки распределения массы темной материи в различных галактиках. Ньютоновская гравитация используется для выполнения вычислений в больших галактических масштабах. Распределение темной материи в различных типах галактик предполагается сферическим без учета сложной структуры галактик, таких как их ядра, внутренние и внешние балджи, диски и гало. Теоретические кривые вращения сопоставлены с наблюдениями для каждой галактики. Посредством алгоритма наименьших квадратов параметры модели выводятся из данных наблюдений и подвергаются Байесовскому информационному критерию, который определяет более предпочтительную модель. Массы темной материи рассчитываются для каждой галактики со всеми перечисленными профилями и сравниваются с видимыми массами галактик. Результаты согласуются с данными в литературе.

Ключевые слова: галактики, кривые вращения, темная материя.

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REFERENCES

[1] Milgrom M. (1983) A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis, The Astrophysical Journal, 270:365-370. DOI: 10.1086/161130

[2] Blioh P.V., Minakov A.A. (1990) Gravitational lenses [Gravitatsionnyye linzy]. Kiev, Russia. ISBN: 5-07-001197-9

[3] Steigman G., Turne M.S. (1985) Cosmological constraints on the properties of weakly interacting massive particles, Nuclear Physics B, 253:375-386. DOI: 10.1016/0550-3213(85)90537-1

[4] Cirelli M, Gennaro C. (2011) PPPC 4 DM ID : A Poor Particle Physicist Cookbook for Dark Matter Indirect Detection, JCAP, 2011. DOI: 10.1088/1475-7516/2011/03/051

[5] Siutsou I., Arguelles C.R., Ruffini R. (2015) Dark Matter Massive Fermions and Einasto Profiles in Galactic Haloes, Astronomy Reports, 59:656-666. DOI: 10.1134/S1063772915070124

[6] Einasto J. (2013) Dark Matter, Brazilian Journal of Physics, 43:369-374. DOI: 10.1007/s13538-013-0147-9

[7] Marchesini D., Elena D'Onghia, Chincarini G., Claudio F. (2002) Hα rotation curves: the soft core question, The Astrophysical Journal 575: 801-813. DOI: 10.1086/341475

[8] Levenberg K. (1944) A method for the solution of certain non-linear problems in least squares, Quarterly of Applied Mathematics, 2:164-168. DOI: 10.1090/qam/10666

[9] Marquardt D. (1963) An algorithm for least-squares estimation of nonlinear parameters, SIAM Journal on applied mathematics, 11: 431-441. DOI: 10.1137/0111030

[10] Schwarz G. (1978) Estimating the Dimension of a Model, Institute of Mathematical Statistics, 6:461-464. DOI: 10.1214/aos/1176344136

[11] Walter F., Brinks E., de Blok, Bigiel F., Kennicutt R., Leroy A. (2008) THINGS: the hi nearby galaxy survey, The Astrophysical Journal, 136: 2563-2647. DOI: 10.1088/0004-6256/136/6/2563

[12] Broeils AH (1992) The mass distribution of the dwarf spiral galaxy NGC 1560, Astronomy and astrophysics, 256:19-32. ISSN: 0004-6361 (it does not have DOI)

[13] Stéphanie C., Carignan C. (1991) A Dark-Halo-Dominated Galaxy: NGC 5585, The Astronomical Journal, 102:904-915. DOI: 10.1086/115922

[14] Sofue Y. (2015) Dark halos of M31 and the Milky Way, Astronomical Society of Japan, 67(4), 75(1-9). DOI: 10.1093/pasj/psv042

[15] Salucci P. (2018) Dark matter in galaxies: evidences and challenges, Foundations of Physics, 48:1517-1537. DOI: 10.1007/s10701-018-0209-5

[16] David A. Buote, Aaron J. Barth. (2019) The extremely high dark matter halo concentration of the relic compact elliptical galaxy Mrk 1216, The Astrophysical Journal, 877:91. DOI: 10.3847/1538-4357/ab1008

[17] Danieli S., Pieter van Dokkum, Conroy C. (2019) Still missing dark matter: KCWI high-resolution stellar kinematics of NGC 1052-DF2, The Astrophysical Journal Letters, 874:12. DOI: 10.3847/2041-8213/ab0e8c

[18] Merle A. (2013) keV Neutrino model building, International Journal of Modern Physics D, 22. DOI: 10.1142/S0218271813300206

[19] Denissyuk E.K., Valiullin R.R. (2019) Rotating curve of the galaxy NGC 1068, News of the National Academy of Sciences of The Republic of Kazakhstan. Physico-Mathematical Series. Vol. 3, No. 325, P. 153 - 157. ISSN 1991-346X. https://doi.org/10.32014/2019.2518-1726.34

[20] Fraternali F., Sancisi R., Tom Oosterloo (2002) Deep HI Survey of the spiral galaxy NGC 2403, The Astronomical Journal, 123: 3124-3140. DOI: 10.1086/340358

[21] Macri L.M., Stetson PB., Bothun G.D., Jha S. (2001) The discovery of cepheids and a new distance to NGC 2841 using the Hubble space telescope, The Astrophysical Journal, 559:243-259. DOI: 10.1086/322395

[22] Joshua J.Adams., Gebhardt K., Hill G.J. (2003) High-resolution measurements of the dark matter halo of NGC 2976: Evidence for a shallow density profile, The Astrophysical Journal, 596: 957-981. DOI: 10.1086/378200

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OPTIMIZATION OF THE SOLID FUEL COMBUSTION PROCESS IN COMBUSTION CHAMBERS IN ORDER TO REDUCE HARMFUL EMISSIONS

Abstract. The methods of numerical simulation are used to study the processes of heat and mass transfer in the combustion chamber of an operating coal-burning Kazakhstan thermal power plant. Computational experiments were performed on the combustion of high-ash Karaganda coal in the combustion chamber of the BKZ-75 boiler (Shakhtinsk, Karaganda region). As a result of numerical simulation of the combustion processes, the distributions of the total velocity vector, temperature fields, concentration fields of nitrogen oxides NO over the entire volume of the combustion chamber and at its exit were obtained. A comparative analysis of the characteristics of heat and mass transfer processes for the two studied modes of supplying fuel to the combustion chamber through burner devices is given for the direct-flow method of supplying the mixture when the burners are located on opposite side walls and the vortex method of supplying the mixture when the burners are installed at a 30-degree angle from the center of symmetry of the boiler. It is shown that the vortex method of supplying air mixtures allows optimizing the combustion of high-ash coal, since in this case there is an increase in temperature in the core of the torch and a decrease in it at the exit from the combustion chamber, which has a significant effect on the chemical processes of the formation of combustion products. The average value of the concentration of nitrogen oxide NO at the outlet of the combustion chamber decreases when using burner devices with a swirl of the mixture flow and conforms to norms the maximum permissible concentration.

Key words. numerical simulation, solid fuel, combustion chamber, direct-flow and vortex methods of supplying air mixtures, velocity, temperature, nitrogen oxides.

Introduction

The development of the fuel and energy complex and energy is one of the most important foundations for the development of all modern material production. Countries with the necessary resources and the ability to develop long-term plans for their use receive undeniable competitive advantages. The issue of choice, operation, and, first of all, the creation of new, highly efficient energy and resource-saving technologies becomes relevant for the heat power industry [1-2].

In this paper, using modern 3D modeling technologies [3-11], a comprehensive study of the thermal processes and aerodynamic characteristics of the combustion chamber of an existing Kazakhstan energy facility (Shakhtinskaya TPP, Kazakhstan) is carried out.. Based on the numerical solution of the system of convective heat and mass transfer equations [12-13], taking into account the kinetics of chemical reactions, two-phase flow, nonlinear effects of convective and radioactive heat transfer, and 3D modeling methods, aerodynamic, thermal, and concentration characteristics are obtained over the entire volume of the combustion chamber, in its main sections and at the exit [14-21].

A comparative analysis of the characteristics of heat and mass transfer processes for the two studied modes of supplying fuel to the combustion chamber through burner devices is given for the direct-flow method of supplying the mixture when the burners are located on opposite side walls and the vortex method of supplying the mixture when the burners are installed at a 30^0 angle from the center of symmetry of the boiler.

The results of the studies allow us to develop appropriate technological solutions for installing burner devices (direct-flow or vortex) in the combustion chamber under study and to optimize the process of burning high-ash Kazakhstan coal in order to minimize harmful emissions into the atmosphere.

2. Description of the combustion chamber for conducting computational experiments

For conducting numerical experiments, the combustion chamber of the BKZ-75 boiler was installed at the Shakhtinskaya TPP (Shakhtinsk, Kazakhstan), in which Karaganda coal with an ash content of 35.1% is burned. Steam boiler BKZ-75 - vertically water-tube, productivity 75 t/h (51.45 Gcal/h) [22-27]. The boiler is equipped with four pulverized coal burners installed in two burners from the front and from the rear in one tier. Figure 1 shows the finite-difference grid for conducting computational experiments and the design of various burner devices (direct-flow and vortex) of the combustion chamber of the BKZ-75 boiler.

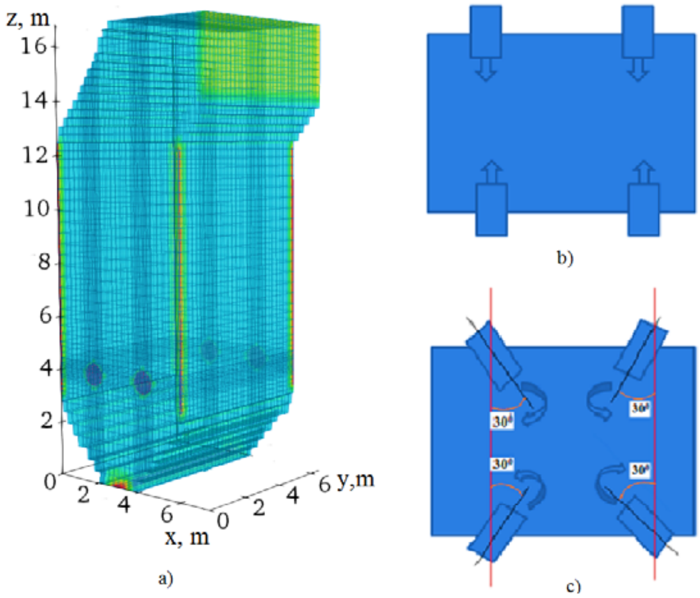


Figure 1 - Finite-difference grid of the combustion chamber of the boiler BKZ-75 of the Shakhtinskaya TPP and designs of burner devices of the combustion chamber of the boiler BKZ-75: a) straight-through burners; b) burners with a spin of the flow of the air mix

To carry out computational experiments in the combustion chamber of the BKZ-75 boiler, two cases were investigated 1) a direct-flow method of supplying air mixtures – burners are located on opposite side walls; 2) the vortex method of supplying the mixture - burners with a swirl angle of the mixture flow and tilting them to the center of symmetry of the boiler by 30^0.

3. Results

This work presents the results of computational experiments, the distribution of the full velocity vector, the temperature and concentration fields of nitrogen oxide NO for two cases of fuel supply to the combustion chamber of the BKZ-75 boiler (direct-flow and vortex). Figure 2 shows the distribution of the total velocity vector in longitudinal sections of the combustion chamber of the BKZ-75 boiler. We see that in the direct-flow method of supplying air mixtures, the flows, colliding in the center at a right angle, are cut in the region of the cold funnel and towards the exit from the combustion chamber, with the formation of a vortex flow of lower intensity. With the vortex method of supplying the air mixture, four swirling flows in the region of the belt of the burner devices guiding from the burners collide with each other in the central part of the combustion chamber at an angle of 30^0. Then, having united in two main streams, they are dissected, forming vortex flows of flow more in the horizontal region of the combustion chamber. High stability of the vortex flow (vortex) position increases the residence time of coal dust in the combustion zone will significantly reduce the formation of the concentration of harmful substances.

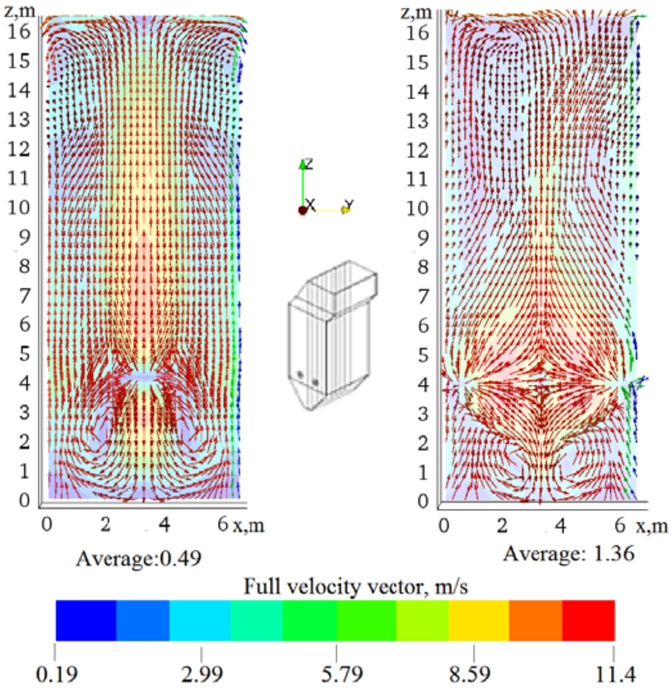


Figure 2. The distribution of the field of the full velocity vector in longitudinal sections (x=3) of the combustion chamber of the boiler BKZ-75:
a) direct-flow method of supplying air mixture; b) vortex method of supplying air mixture

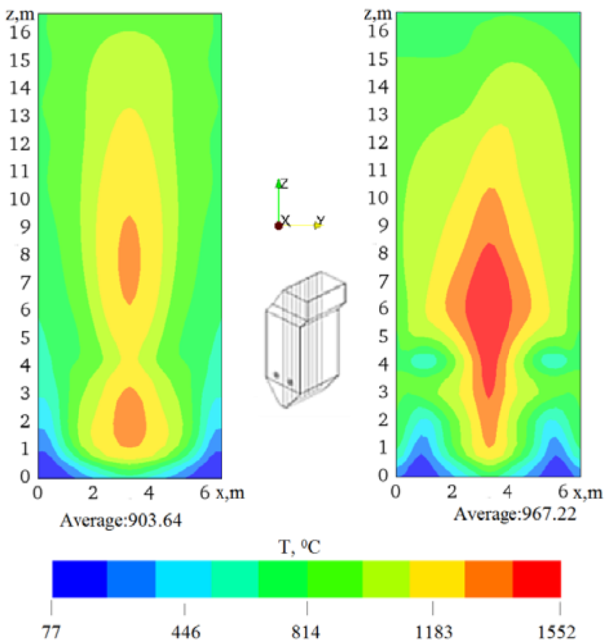


Figure 3 - The distribution of the field of the temperature in longitudinal sections (x=3) of the combustion chamber of the boiler BKZ-75:
a) direct-flow method of supplying air mixture; b) vortex method of supplying air mixture

Figure 3 illustrates the temperature field in the longitudinal sections of the combustion chamber of the BKZ-75 boiler for the two studied modes of supply of air mixture (direct-flow and vortex). We see that the temperature has maximum values in an area close to the location of the burner devices. With the direct-flow method of supplying the air mixture, the average temperature in the longitudinal section (x=3) of the combustion chamber of the BKZ-75 boiler is 903.64ºC, and in the case of a vortex feed of a mixture, the temperature value increases and amounts to 967.22ºC. This is due to the vortex nature of the flow, providing maximum convective transport and an increase in the residence time of coal particles in the combustion chamber of the BKZ-75 boiler.

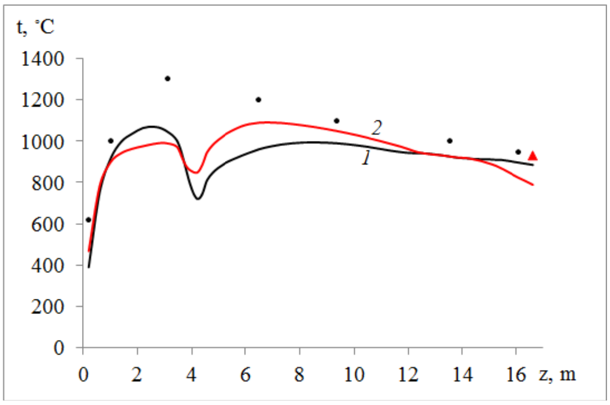
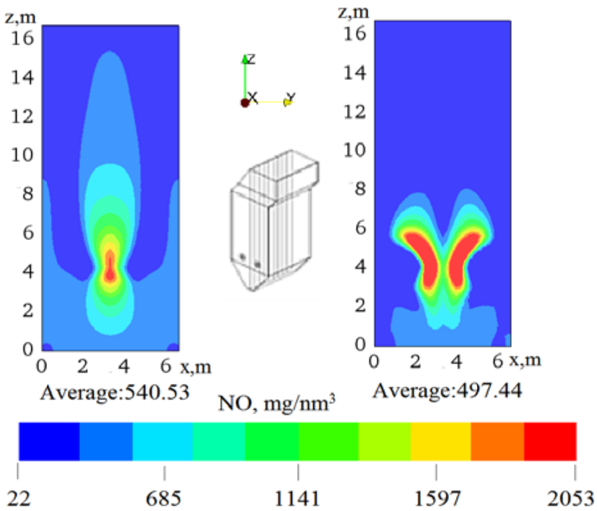


Figure 4 - Temperature distribution along the height of the combustion chamber the BKZ-75 boiler with: 1– direct-flow method of supplying air mixture; 2 – vortex method of supplying air mixture; ● - experimental data at TPPs [28]; ▲ – is theoretical values obtained by the method of thermal calculation (CBTI – Central Boiler-and-Turbine Institute) [29]

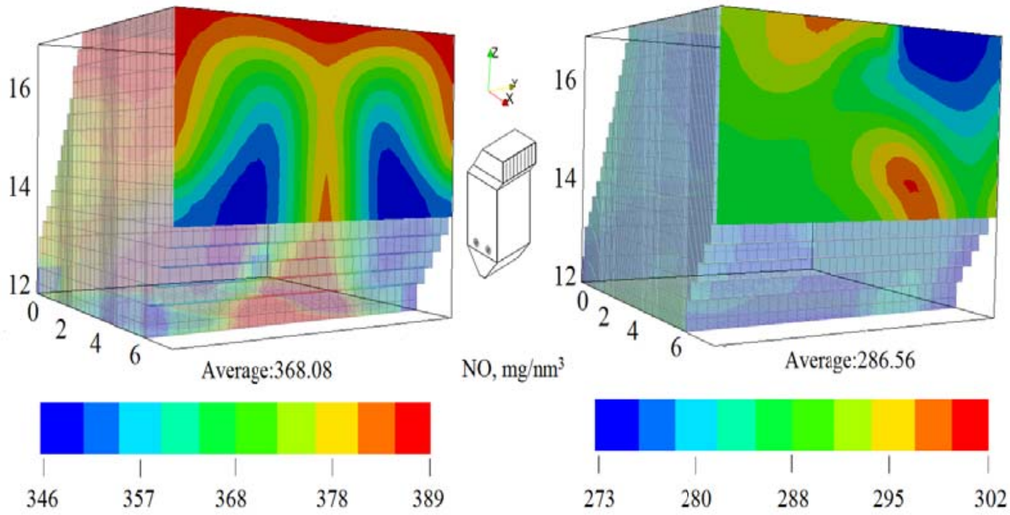
Figure 4 shows a comparative analysis of the distribution of the average temperature in the cross section over the height of the combustion chamber for the two studied modes of supply of air mixture (direct-flow and vortex). In the case of a vortex feed of an air mixture, an increase in the extent of the zone of maximum temperatures is observed. The minimum in the curves associated with the low temperature of the air mixture entering the combustion chamber through the burners. An increase in the temperature in the core of the flame and a decrease in its output exerts a significant effect on the chemical processes of the formation of combustion products. The temperature at the outlet of the combustion chamber is confirmed by its theoretical value calculated by the CBTI method for direct-flow supply of air mixture [29].

Distributions of nitrogen oxide NO concentrations in different sections of the combustion chamber are presented in the Figures 5-6.



a) b)

Figure 5 - Distribution of nitrogen oxide NO area in longitudinal sections (x=3) of the combustion chamber the BKZ-75 boiler with:
a) direct-flow method of supplying air mixture; b) vortex method of supplying air mixture



a) b)

Figure 6 - Distribution of nitrogen oxide NO at the outlet of the combustion chamber the boiler BKZ-75 with: a) direct-flow method of supplying air mixture; b) vortex method of supplying air mixture

As can be seen from the figures, the zone of maximum formation of nitrogen oxide NO is the region of high temperatures and intense vortex flow. Intensive mixing of fuel and oxidizing agent, created by turbulent flows of injected air mixture near the burners, as well as high temperature in the torch core, create favorable conditions for the formation of nitrogen oxides. The average value of the concentrations of nitrogen oxide NO in this region is equal to the direct-flow method of supplying air mixture – 540.53 mg/nm^3 (Figure 5a), and with a vortex method of supplying air mixtures – 497.44 mg/nm^3 (Figure 5b).

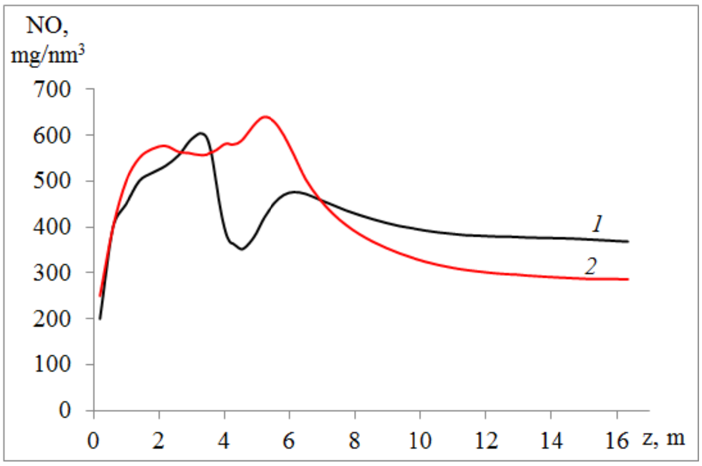


Figure 7 - Distribution the concentration of nitrogen oxide NO at the outlet of the combustion chamber the boiler BKZ-75 with: 1 - direct-flow method of supplying air mixture nitrogen oxide NO; 2 - vortex method of supplying air mixture nitrogen oxide NO

However, towards the exit from the combustion chamber (Figures 6-7), a uniform decrease in the NO concentration is observed, since this region contains less oxygen and a fuel component. In addition, in the case of using burner devices with swirling of the mixture flow, the temperature along the height of the combustion chamber monotonously decreases, as a result of which the rate of formation of nitrogen oxide NO. At the exit from the combustion chamber, the average value of the concentration of nitrogen oxide NO with a direct-flow method of supplying air mixture is 368.08 mg/nm^3 (Figure 6a and 7a, curve 1), and with vortex burner devices – 286.56 mg/nm^3 (Figure 6b and 7b, curve 2), that on 81 mg/nm^3 less.

The results indicate the advantages of choosing a vortex method of supplying air mixtures to optimize the combustion of high-ash coal in the combustions of power plants and reduce harmful dust and gas emissions into the environment.

Conclusion

The methods of numerical simulation are used to study the processes of heat and mass transfer in the combustion chamber of an active coal-burning Kazakhstan TPP. Computational experiments were performed on the combustion of high-ash Karaganda coal in the combustion chamber of the BKZ-75 boiler (Shakhtinsk, Karaganda region).

As a result of the numerical simulation of the combustion processes during the combustion of high-ash coal, the distributions of the total velocity vector, temperature fields, concentration fields of nitrogen oxides NO over the entire volume of the combustion chamber of the boiler and at its exit were obtained.

A comparative analysis of the characteristics of heat and mass transfer processes for the two studied modes of supplying fuel to the combustion chamber through burners with a direct-flow method of supplying the mixture when the burners are located on opposite side walls and a vortex method (swirling flow) of the mixture supplying when the burners are installed with their slope to the center of symmetry of the boiler is presented 30^0.

The vortex method of supplying the mixture allows to optimize the combustion of high-ash coal, due to the circulation movement, the residence time of the fuel particles in the combustion chamber increases, there is an increase in temperature in the flame core and its decrease at the exit of the combustion chamber, which has a significant effect on the chemical processes of the formation of combustion products. In this case, the average value of the concentration of nitrogen oxide NO when using burner devices with a swirl of the mixture flow at the outlet of the combustion chamber decreases and corresponds to the norms of maximum permissible concentration.

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ЗИЯНДЫ ҚАЛДЫҚТАРДЫ АЗАЙТУ МАҚСАТЫНДА ЖАНУ КАМЕРАСЫНДА ҚАТТЫ ОТЫНДЫ ЖАҒУ ПРОЦЕСІН ОҢТАЙЛАНДЫРУ

Аннотация. ЖЭО-ның қазақстандық қолданыстағы көмір жағатын жану камерасында жылу және масса алмасуын сандық модельдеу әдістерімен зерттелді. БКЗ-75 (Шахтинск, Қарағанды облысы) қазандығының жану камерасында жоғары күлді Қарағанды көмірін жағу кезінде есептеу тәжірибелері жүргізілді. Жану процестерін сандық модельдеудің нәтижесінде мыналар алынды: жылдамдық векторының толық таралуы, температуралық өріс, NO азот оксидтерінің концентрациялық өрістерінің жану камерасының барлық көлемінде және оның шығысында. Жану камерасына оттық құрылғылар арқылы жанармай жеткізудің екі режимі жылу және масса алмасу процестерінің сипаттамаларына салыстырмалы үшін зерттеліп талдау ұсынылған: оттықтар қарама-қарсы жақ қабырғаларында орналасқан кезде ауа қоспасын берудің тікелей әдісі және қыздырғыштар өздерінің градусымен 30 градусқа бейімделген күйде орнатылған кезде ауа қоспасын берудің құйынды әдісі ұсынылған. Құйынды ауа қоспасын беру әдісі жоғары күлді көмірдің жануын оңтайландыруға мүмкіндік береді, өйткені бұл жағдайда алау өзегіндегі температураның жоғарылауы және жану камерасынан шыққан кезде оның төмендеуі байқалады, жану өнімдерінің пайда болуының химиялық процестеріне айтарлықтай әсер етеді. Азот оксиді NO концентрациясының орташа мәні жану камерасының шығысындағы қоспаның ағынының серпілісімен қыздырғыш құрылғыларды пайдалану кезінде төмендейді және ПӘК сәйкес келеді.

Түйін сөздер. Сандық моделдеу, қатты отын, жану камерасы, ауа қоспаларын берудің тікелей ағынды және құйынды әдістері, жылдамдық, температура, азот оксидтері

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ОПТИМИЗАЦИЯ ПРОЦЕССА СЖИГАНИЯ ТВЕРДОГО ТОПЛИВА В ТОПОЧНЫХ КАМЕРАХ С ЦЕЛЬЮ СНИЖЕНИЯ ВРЕДНЫХ ВЫБРОСОВ

Аннотация. Методами численного моделирования исследованы процессы тепломассопереноса в топочной камере действующей углесжигающей казахстанской ТЭЦ. Выполнены вычислительные эксперименты по сжиганию высокозольного карагандинского угля в камере сгорания котла БКЗ-75 (г. Шахтинск, Карагандинская область). В результате проведения численного моделирования топочных процессов были получены: распределения вектора полной скорости, температурные поля, поля концентраций оксидов азота NO по всему объему топочной камеры и на выходе из нее. Представлен сравнительный анализ характеристик процессов тепломассопереноса для двух исследуемых режимов подачи топлива в камеру сгорания через горелочные устройства: прямоточный способ подачи аэросмеси, когда горелки расположены на противоположных боковых стенках и вихревой способ подачи аэросмеси, когда горелки установлены с наклоном их к центру симметрии котла на 30 градусов. Показано, что вихревой способ подачи аэросмеси позволяет оптимизировать процесс сжигания высокозольного угля, поскольку в этом случае наблюдается увеличение температуры в ядре факела и снижение ее на выходе из камеры сгорания, что оказывает существенное влияние на химические процессы образования продуктов горения. Среднее значение концентрации оксида азота NO на выходе из топочной камеры уменьшается при использовании горелочных устройств с закруткой потока аэросмеси и соответствует нормам ПДК.

Ключевые слова. Численное моделирование, твердое топливо, камера сгорания, прямоточный и вихревой способы подачи аэросмеси, скорость, температура, оксиды азота.

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REFERENCES

[1] Georgiev A, Baizhuma ZhT, Manatbayev RK, etc (**2018**) The use of a new "clean" technology for burning low-grade coal in on boilers of Kazakhstan TPPs. Bulgarian Chemical Communications, 50(G):53 – 60.

[2] Messerle VE, Ustimenko AB, etc (**2016**) Reduction of noxious substance emissions at the pulverized fuel combustion in the combustor of the BKZ-160 boiler of the Almaty heat electro power station using the "Overfire Air" technology. Journal Thermophysics and Aeromechanics, 23(1):125-134. DOI: 10.1134/S0869864316010133.

[3] Bolegenova SA, Beketayeva MT, Askarova A, etc (**2018**) Modeling of heat and mass transfer in high-temperature reacting flows with combustion. Journal High Temperature, 56(5):738-743. DOI: 10.1134/S0018151X1805005X.

[4] Gabitova Z, Leithner R, Ergalieva A, etc (**2016**) Computational modeling of heat and mass transfer processes in combustion chamber at power plant of Kazakhstan. Proceedings MATEC Web of Conferences, 76:UNSP06001 DOI:10.1051/matecconf/20167606001.

[5] Shortanbayeva ZhK, Maksutkhanova AM, Aldiyarova AN, etc (**2016**) Mathematical modeling of heat and mass transfer in the presence of physical-chemical processes. Proceeding of 15th international scientific conference on renewable energy and Innovate Technologies, Tech Coll Smolyan, Smolyan, Bulgaria, Bulgarian Chemical Communications, 48(E):272-277.

[6] Baktybekov KS, Pilipenko NV, Shortanbayeva ZhK, etc (**2018**) Investigation of the different Reynolds numbers influence on the atomization and combustion processes of liquid fuel. Bulgarian Chemical Communications, 50(G):68 - 77

[7] Beisenov KhI, Maxutkhanova AM, Turbekova AG, etc (**2017**) A computational experiment for studying the combustion of thermochemically-gasified coal in the combustion chamber of the boiler BKZ-160. News of the national academy of sciences of the Republic of Kazakhstan-series physico-mathematical, 2(312):75-80 DOI: 10.1515/eng-2018-0020.

[8] Safarik P, Askarova AS, Beketayeva MT, etc (**2018**) Modern computing experiments on pulverized coal combustion processes in boiler furnaces, News of the national academy of sciences of the Republic of Kazakhstan-series Physico-mathematical, 6(322): 5-14 DOI: 10.32014/2018.2518-1726.11.

[9] Boranbayeva A, Maxutkhanova A, Shortanbayeva Zh, etc (**2016**) Application of 3D modelling for solving the problem of combustion coal-dust flame. Proceeding of 15th international scientific conference on renewable energy and Innovate Technologies, Tech Coll Smolyan, Smolyan, Bulgaria, Bulgarian Chemical Communications, 48(E):236-241.

[10] Yergaliyeva A, Shortanbayeva Zh, Gabitova Z, etc (**2017**) Simulation of the aerodynamics and combustion of a turbulent pulverized-coal flame. Proceedings of 4th International Conference on Mathematics and Computers in Sciences and in Industry (MCSI 2017). Corfu Island, Greece. P.92-97. DOI: 10.1109/MCSI.2017.23.

[11] Bolegenova S, Askarova A, Shortanbaeva ZhK, etc (**2017**) Numerical modeling of burning pulverized coal in the combustion chamber of the boiler PK 39. News of the national academy of sciences of the Republic of Kazakhstan-series physico-mathematical, 2(312):58-63.

[12] Askarova AS, Safarik P, Nugymanova AO, etc **(2019)** Simulation of low-grade coal combustion in real chambers of energy objects. Journal Acta Polytechnica, 59(2):98-108. doi.org/10.14311/AP.2019.59.0098

[13] Muller H. **(1992)** Numerische Berechnung Dreidimensionaler Turbulenter Strömungen in Dampferzeugern Mit Wärmeübergang und Chemischen Reaktionen am Beispiel des SNCR–Verfahrens und der Kohleverbrennung. Fortschritt–Berichte VDI-Verlag. 6(268): 158 (in German).

[14] Askarowa A, Buchmann M, (**1997**) Structure of the flame of fluidized-bed burners and combustion processes of high-ash coal. Proceedings of 18th Dutch-German Conference on Flames, VDI Berichte, 1313:241-244.

[15] Maximov VYu, Bolegenova SA, etc **(2012)** Mathematical simulation of pulverized coal in combustion chamber. Proceedings of 20th International Congress of Chemical and Process Engineering, Prague, Czech Republic. 42:1150-1156.

[16] Chtab-Desportes A, Gorokhovski M, Voloshina I, etc (**2010**) Stochastic simulation of the spray formation assisted by a high pressure. AIP Conference Proceedings, 1207:66-73

[17] Beketayeva M, Bekmuhamet A, Ergalieva A, etc (**2016**) 3D modeling of heat and mass transfer during combustion of solid fuel in BKZ-420-140-7C combustion chamber of Kazakhstan. Journal of Applied Fluid Mechanics, 9(2):699-709.

[18] Heierle EI, Manatbayev RK, Yergaliyeva AB, etc (**2016**) CFD study of harmful substances production in coal-fired

power plant of Kazakhstan. Proceedings of 15th International Scientific Conference on Renewable Energy and Innovative Technologies: Tech Coll Smolyan, Smolyan, Bulgaria, Bulgarian Chemical Communications, 48(E2):260-265.

[19] Ustimenko AB, Messerle VE, Karpenko EI, etc (**2007**) Mathematical modeling of the processes of solid fuel ignition and combustion at combustors of the power boilers. Proceedings of the 7th International Fall Seminar on Propellants, Explosives and Pyrotechnics, Xian, 7:672-683.

[20] Askarova AS, Bolegenova SA, Ospanova ShS, etc (**2017**) Investigation of aerodynamics and heat and mass transfer in the combustion chambers of the boilers PK-39 and BKZ-160. News of the national academy of sciences of the Republic of Kazakhstan-series physico-mathematical, 2(312):27-38.

[21] Leithner R, Vockrodt S, Schiller A, (**1999**) Firing technique measures for increased efficiency and minimization of toxic emissions in Kasakh coal firing. Proceedings of 19th German Conference on Flames, Dresden, Germany VDI Gesell Energietechn; Verein Deutsch Ingn.Combustion and incineration, 1492:93-97. (in Eng).

[22] Ospanova ShS, Bekmuhamet A, etc (**2012**) Numerical research of aerodynamic characteristics of combustion chamber BKZ-75 mining thermal power station. Procedia Engineering, 42:1250-1259. DOI: 10.1016/j.proeng.2012.07.517.

[23] Ospanova Sh, Manatbayev R, Mazhrenova N, etc (**2016**) 3D modelling of heat and mass transfer processes during the combustion of liquid fuel, Proceedings of 15th International Scientific Conference on Renewable Energy and Innovative Technologies, Tech Coll Smolyan, Smolyan, Bulgaria, Bulgarian Chemical Communications, 48(E):229-235.

[24] Bekmukhamet A., Beketayeva M, Gabitova Z, etc (**2014**) Control harmful emissions concentration into the atmosphere of megacities of Kazakhstan Republic. International conference on Future Information Engineering, Beijing, Peoples China, 10:252-258. DOI: 10.1016/j.ieri.2014.09.085.

[25] Beketayeva MT, Ospanova ShS, Manatbayev RK, etc (**2018**) The computational study of heat and mass transfer processes at combustion of pulverized kazakh coal in real conditions of energy objects. Journal Bulgarian Chemical Communications, 50(G):61- 67.

[26] Safarik P, Maximov VYu, Bolegenova SA, etc **(2019)** 3D modeling of combustion thermochemical activated fuel. News of the national academy of sciences of the Republic of Kazakhstan-series physico-mathematical, 2(324):9-16. DOI: 10.32014/2019.2518-1726.7

[27] Mamedova MR, Mazhrenova NR, Bolegenova SA, etc **(2019)** Computational experiments for research of flow aerodynamics and turbulent characteristics of solid fuel combustion process. News of the national academy of sciences of the Republic of Kazakhstan-series physico-mathematical, 2(324):46–52. DOI: 10.32014/2019.2518-1726.11

[28] Teplovoy raschet kotlov (normativnyiy metod). **(1998)** SPb.:AOOT "NPO TsKTI".

[29] Alijarov BK, Alijarova MB (**2012**) Szhiganie kazahstanskih uglej na TJeS i na krupnyh kotel'nyh: opyt i perspektivy. RGP Gylym ordasy, Kazakhstan.

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FERMION THEORY OF COLLECTIVE STATES OF NUCLEI, ITS APPLICATION TO THE STRUCTURE OF REAL SYSTEMS

Abstract. Based on the nucleon-pair shell model, in which the fermionic space is cut by the "realistic" SD-operators by the generalized senority method, the microscopic structure of the collective states of the nuclei of the average atomic weight is studied. In this case, the effects of splitting of single-particle levels on the collective-pair structure of the system are taken into account. To solve such a multiparticle problem, we use the generalized quasispin method and double tensors, which facilitate the calculation of the matrix elements of pair interactions of nucleons. The total Hamiltonian is diagonalized exactly in fermionic space without applying the procedure for mapping fermion operators into bosonic operators. The parameters of the interacting boson model are calculated on the basis of the permuted fermion approach. The theory is applied to the study of the properties of the collective states of even isotopes of ruthenium with N = 58-66. The spectrum of low-energy states is also calculated for the probabilities of E2 transitions in them and they are compared with the available experimental data.

I. Introduction

The low-energy collective states of the nuclei of medium and heavy atomic nuclei are well described by the interacting boson model (IBM) [1-4]. The parameters of such a phenomenological theory are usually chosen from comparison with experimental data, and they smoothly change with an increase in the number of nucleons in the isotopes of systems.

On the other hand, the observed changes in the parameters of the IBM model as a function of N and Z are in good agreement with the first approximation of the approach that takes into account the severity schemes in interacting fermion systems [5-7].Attempts were also made to substantiate the IBM by calculating model parameters from detailed microscopic approaches. The description of collective states in terms of fermion degrees of freedom is an interesting and important problem in the theory of nuclear structure.But because of the difficulty of carrying out numerical calculations in a huge shell-model space for nuclei with a sufficiently large number of particles, we have to use some types of truncating schemes for the fermion space.In many cases, the so-called SD –pair circumcision of the enormous fermion-shell space is used, however, difficulties remain regarding the application of “realistic” SD –nucleated pairs as building blocks in the model [8-11].To overcome such obstacles in the microscopic calculations, the parameters of the IBMare determined by the method of mapping the collective shell space into the bosonic one, and then systematic calculations are carried out in the bosonic space. The Ohtsuka-Arima-Yakello (OAY) map [12,13] based on the generalized seniority scheme was used most of all [14].

In this work, we use the nucleon – pair shell model in which the fermion space is cut off by ″ realistic ″ SD – pair operators taking into account the effect of splitting of single-particle levels into a collective – pair structure of the system.Also here, the full Hamiltonian is diagonalized exactly in the fermion space with generalized senority, without using the mapping procedure, which gives a fermion pattern of collective excitations of nuclear states. The used approach to the study of collective states goes into the so-called fermion-dynamic symmetric model (FDSM) [15-24], in the case of neglecting splittings of one-particle states that affect the collectivity of levels.

In this work, systematic calculations of the parameters of the model of interacting bosons are carried out, as applied to the study of the properties of low-energy states of even Ru isotopes with neutron numbers N = 58-66. Selection of the nuclei due to the fact that, firstly, they generalized quantum number seniority (quasi- spin generalized) is most purely secondly due to prostate configurations of excited nuclear states application of this method is simple and clear. In addition to the spectra of nuclei, the behavior of the wave functions of states is also discussed by calculating the ratios B (E2) and δ (E2 / M1) for lower states. The obtained values are compared with known experimental data.

II.Generalized quasispin space and pair interaction of fermions

The formalism of the generalized quasi-spin (generalized senority) [3, 4] is used, which allows one to exactly solve a multi-pair fermion problem with a fixed number of particles with given internuclone forces.It introduces a double tensor in the usual and generalized seniority spaces, by means of which the eigenfunctions and eigenvalues of numerous pairwise operators are easily found.Generalized quasispin fermion space we introduce via operators:

$$S_{+}= \sum_j \alpha_j S_+^j, S_{-}= \sum_j \frac{1}{\alpha_j} S_-^j, \quad S_{0}= \sum_j S_0^j = {(N-\Omega)}/{2}, \tag{2.1}$$

in which α_j – are coefficients reflecting the amplitudes of the probability of population of the orbit j and they are normalized:

$$\Omega = \sum_j \alpha_j \Omega_j = \sum \alpha_j (j + {1}/{2}).$$

The quasi-spin operators of the shell configuration j^n, satisfy the usual commutation relations, which the angular momentum operators obey:

$$S_+^j = \sqrt{\Omega} A^+(jj00) = \sqrt{\Omega} \left(jm_1 jm_2 \middle| 00 \right\rangle a_{jm_1}^+ a_{jm_2}^+ , S_-^j = \sqrt{\Omega} \tilde{A}(jj00),$$

$$S_0^j = {1}/{2} \left(N^j - \Omega^j \right) = \frac{1}{2} \left(\sum a_{jm}^+ \tilde{a}_{j-m} (-)^{j-m} - \Omega^j \right). \tag{2.2}$$

In addition, double tensors are introduced in the spaces of both quasispins with the moment λ, as well as in the usual with tensors of rank k and their corresponding Z-projections s and q.For the case when k is even, they are written in the form:

$$T_{1,q}^{(1,k)}(jj) = A^+(jjkq), T_{1,q}^{(1,k)}(jj) = \tilde{A}(jjkq),$$

$$T_{0,q}^{(1,k)}(jj) = U(jjkq) + \sqrt{{\Omega}/{2}}\, \delta(k,0). \tag{2.3}$$

Here they are double tensors; in the usual space of rank k, and simultaneously a tensor of rank 1 in a quasispin space. Any single-particle operator is proportional to the double tensor of the first rank in a quasispin space, and in the ordinary one it is proportional to the k-rank tensor $T^{(1,k)}(jj)$.

The reduced matrix element of the single-particle operator of n particles is written through a matrix of single particles with senority ϑ.

$$\langle j^n \vartheta \alpha J || \sum_j f_i^n || j^n \vartheta \alpha' J' \rangle = \frac{\Omega - n}{\Omega - \vartheta} \sqrt{\frac{(n-\vartheta+2)(2\Omega-n-\vartheta+2)}{4(\Omega-\vartheta+1)}} \langle j^\vartheta \vartheta \alpha J \left\| \sum_j f_i^k \right\| j^\vartheta \vartheta - 2, \alpha' J' \rangle. \tag{2.4}$$

A completely similar method can be used to simplify the calculation of two-particle matrix elements using similar reduction formulas. Usingthedoubletensors (2.3), thepairinginteraction operator is written:

$$V = \sum_{j_1 j_2 j_3} \sqrt{2J+1}\, G_1(j_1 j_2 j_3 j_4) [A^+(j_1 j_2) \times \tilde{A}(j_3 j_4)]_0^{(0)} \tag{2.5}$$

$$G_J = {(1 + \delta j_1 j_2)(1 + \delta j_3 j_4)}/{4} \cdot \langle j_1 j_2 J | V | j_3 j_4 J \rangle,$$

can also be expressed in terms of double tensor

$$V = -\sum_J \sqrt{2J+1}\, G_J \left[T_{jj}^{(1,J)} \times T_{jj}^{(1,J)}\right]_{00}^{(\lambda,0)} (111-1|\lambda 0) = T^{(0)} + T^{(1)} + T^{(2)} \qquad (2.6)$$

For example, $T^{(\lambda)} = -\frac{1}{2}\sum_{JM}(-)^{J-M} G_J \left[T_{1M}^{(1,J)} - T_{-1-M}^{(1,J)}\right] = (N-\Omega)F_0$ (2.7)

Here, for example, for $= 0$: $F_0 = -\frac{1}{2\Omega}\sum_J (2J+1)G_J$.

In additiononly$T^{(2)}$) can change senority ϑ by ϑ′ = ϑ + 2, ϑ + 4, then according to the Wigner-Eckart theorem the reduction formula follows:

$$< j^n \vartheta\alpha J|V|j^n \vartheta'\alpha' J' > = \frac{f_2(n)}{f_2(\vartheta)} < j^2 \vartheta\alpha J\left|T^{(2)}\right|j^2 \vartheta'\alpha' J' > \qquad (2.8)$$

where $f_2(n)$=($\frac{1}{2}(\Omega-\vartheta')2\frac{1}{2}(n-\Omega)0|\frac{1}{2}(\Omega-\vartheta')\frac{1}{2}(n-\Omega)$)are the Clebsch-Gordan coefficients.

Matrix elements that are diagonal in senority include the contributions of all three tensors. $T^{(0),(1),(2)}$, which are discussed in detail in the works[4,9].

Operators of generalized quasi-spin(2.1) obey also the usual commutation ratios, however, remain non-Hermitian:$(S_-)^+ \neq S_+$.

$$[S_+, S_-] = 2S_0\,,\ [S_0, S_\pm] = \pm 2S_\mp \qquad (2.9)$$

All Lie groups (2.1) for all values of $\alpha_j$are isomorphic to each other. Therefore, for any set of operators (2.1), we can introduce the complete generalized quasispin operator: $S^2 = S_+S_- + S_0^2 - S_0$.

The state vectors of the quasi-spin operators S and $S_0$are determined by the quantum numbers s and s_0, which are associated with the quantum number seniority and the total number of nucleons N in the form:

$$S = \frac{1}{2}(\Omega-\vartheta) and S_0 = \frac{1}{2}(N-\Omega). \qquad (2.10)$$

Then using commutation relations between $S_\pm$, S_0we have:

$$S_-|s\ \ ,s_{0,}q >= const|s, s_0 - 1, q >,$$
$$S_+|s, s_{0,}q >= const|s, s_0 + 1, q >, \qquad (2.11)$$
$$S_-|s, s_{0,} = -s, q >=0.$$

Thus, according to the quantum number of the generalized quasi-spin s, it is possible to classify the states of the system by the ratio of the rotation of the system in the quasispin space.Therefore, this method is one of the easiest ways to solve many-particle problems.Many-particle matrix elements are expressed in terms of two-particle with the help of reduction formulas and commutation relations between tensors.For example, Hamiltonian pairing interaction

$$H_S = \varepsilon N - GS_+S_- \qquad (2.11)$$

is diagonal in the representation of a generalized seniority.

Wave functions of system states with quantum numbers$\left|s, JM\right\rangle$, therefore, are expressed in form:

$$|s, s_0 JM >= \left\{\frac{(\Omega-\vartheta-n)!}{n!(\Omega-\vartheta)!}\right\}^{\frac{1}{2}} (S_+^1)^n |s, -s, JM > \qquad (2.12)$$

wheren $= (\mathrm{N} - \vartheta)/2$ -the number of paired particles.

Now we consider a many-particle problem in the space of a generalized quasispin with an arbitrary pair interaction operator. The full Hamiltonian, in this case, is conveniently divided into two parts, selecting from it the pairing interaction Hs in the generalized quasispin representation: $H = H_s + W$,where W-operator, expressing the rest of the interaction of particles, but diagonal in the representation of generalized quasispin s

$$W = \sum_{j_1 j_2 j_3 j_4} \left\langle j_1 j_2 \left| V^{'} \right| j_3 j_4 \right\rangle A_+ (j_1 j_2 JM) A_- (j_3 j_4 JM). \quad (2.13)$$

Then the eigenvalue problem for the complete Hamiltonian H, which is diagonal in the s-representation, reduces to solving the equation:

$$H \left| s, S_0 \right\rangle = E(n = s + s_0, v = \Omega - 2s, q) \left| s, s_0 q \right\rangle_0. \quad (2.14)$$

The total energy of the system is also divided into two parts.

$$E(n,v,q) = E(n = 2n + v, v) + E^{'}(n,v,q), \quad (2.15)$$

where, Es is the eigenvalues of the pairing part of the Hamiltonian Hs.

Let us find the conditions under which the full Hamiltonian H is diagonal in the representation of a generalized quasispin. For this, it is necessary that functions (2.14) be eigenfunctions of the operator W:

$$W \left| s, s_0, q \right\rangle = E^{'}(n,v,q) \left| s, s_0, q \right\rangle. \quad (2.16)$$

This equation can be reduced to several easily solvable, independent of n equations. Forthispurpose, considerthecommutator:

$$[W, S_+] = 2 \sum \left\langle j_1 j_2 \left| V^{'} \right| j_3 j_4 \right\rangle A_+ (j_1 j_2 JM) \left\{ \sqrt{\Omega^{j_3}} \alpha j_3 \delta_{JM}^{00} / \sqrt{2} - (-)^{J-M} \alpha j_3 T_{-M}^{J} (j_3 j_4) + (-)^{j_3 + j_4 - M} \alpha j_4 T_{-M}^{J} (j_3 j_4) \right\}$$

$$T_M^J (jj^{'}) = \left(2\sqrt{1 + \delta_{jj^{'}}}\right)^2 \sum_{mm^{'}} (jjm - m^{'} | JM) a_{jm}^{+} \widetilde{a}_{j^{'}m^{'}}, \quad (2.17)$$

where, single-particle operator satisfying the relations:

$$\left[T_M^J (jj^{'}), S_+^{''}\right] = 2\delta_{jj^{'}} A_+ (jj^{'} JM). \quad (2.18)$$

This operator breaks a pair of particles in the $S_+ \left| 0 \right\rangle$ state and puts them into the excited state $A_+ (jj^{'} JM) \left| 0 \right\rangle$.In addition, we introduce the operator of the creation of unpaired particles v with a common angular momentum J:

$$Q^+ (v, JM) \left| 0 \right\rangle = \sum_j \gamma_j^{v,J} Q^+ (j^v, JM) \left| 0 \right\rangle. \quad (2.19)$$

From the normalization condition of the wave functions, we have: $\sum_j \left(\gamma_j^{v,J}\right)^2 = 1$.

Then equation (2.16) can be rewritten in the form:

$$W (S_+)^n Q^+ (v, JM) \left| 0 \right\rangle = E^{'}(n,v,q) Q^+ (v, JM) \left| 0 \right\rangle. \quad (2.20)$$

Expression (2.19) will be satisfied, if only the equalities hold:

$$[[W, S_+] Q^+ (v, JM)] = \lambda(v, J) S_+ Q^+ (v, JM),$$

$$W Q^+ (v, JM) \left| 0 \right\rangle = E^{'}(n,v,q) Q^+ (v, JM) \left| 0 \right\rangle,$$

$$E^{'}(n,v,q) = E^{'}(n,v,q) + n\lambda(v, J).$$

As a result, the condition of diagonalization of the complete Hamiltonian $H = H_s + W$ in the s-representation reduces to solving the system of equations

$$\left.\begin{aligned} &HS_{+}|0\rangle = E_0 S_{+}|0\rangle, \\ &[[H,S_{+}]S_{+}] = 2G(S_{+})^2, \\ &HQ^{+}(v,JM)|0\rangle = E(v,J)Q^{+}(v,JM)|0\rangle, \\ &[[H,S_{+}],Q^{+}(v,JM)] = (vG + \lambda(v,J))S_{+}Q^{+}(v,JM), \end{aligned}\right\} \qquad (2.22)$$

where, $E_{+} = S_{s}(N = 2, v = 0 = 2\varepsilon - G\Omega.)$.

The hollow energy of the system is determined by equality (2.15). In the case of the invariance of W with respect to its rotation in the s-representation, the equality $\lambda(v, J) = 0$ must be fulfilled.

Thus, the solution of the problem with the full Hamiltonian H leads to the lifting of the degeneracy of levels by the angular momentum J in multiplets characterized by a quantum number of generalized senority v whose positions linearly depend on the number of pairs in the system.

III. Fermion structure of collective states of even ruthenium isotopes

The stated microscopic method for calculating the collective state of nuclei is applied to the study of low-energy states of even ruthenium isotopes with atomic weights A = 100-106.

The single-particle functions of a symmetric harmonic oscillator are taken as the basis of the calculations. The potential of the nucleon-nucleon interaction is selected in the form:

$$V = (U_w + U_S\pi_S + U_T S_{12}) f(r, r_0) + U_C \qquad (3.1)$$

where,U_w, U_S, U_T–parameters ofwigner, singlet and tensor forces,$\pi_S$and S_{12}- operators singlet and tensor projection n.Radial force dependence$f(r, r_0)$selected as Gauss potential,U_C -Coulomb potential. The full potential of the pair interaction of nucleons is equal to:

$$V = V_{nn} + V_{pp} + V_{np}$$

As the core of this nuclei in protons and neutrons states is taken the low energy states of the strontium nucleus with Z = 38 and N = 50 [10].The proton shell of the strontium nucleus is completed by filling the $2P_{3/2}$level.

The distance between this overhead and free shell $2P_{1/2}$ of order 3 MeV. Then the shell $2P_{3/2}$can be consideredsemimagic. As a single-particle proton states can be taken, the lower state ^{89}Y: the binding energy of which is:$\varepsilon_{p_{1/2}} = -7{,}07$ MeV, $\varepsilon_{g_{9/2}} = -6{,}16$MeV.

As single-particle neutron states, taken thehole states of $^{131}_{50}Sn$ in MeV:

$$\varepsilon_n = 0(d_{3/2}), 0{,}24(h_{11/2}), 0{,}33(s_{1/2}), 1{,}66(d_{3/2}), 2{,}34(g_{7/2}). \qquad [11]$$

With the above experimental data on single-particle states of near magic nuclei Z = 38, N = 50realizethe corresponding choices of the parameters of paired interactions of valence nucleons.The parameters of pair interactions of nucleons are determined from the description of experimental spectra of even Ru isotopes.

The depth of the proton-proton interaction Vpp should vary slowly in nuclear isotopes, but differ from each other by a small amount. In this paper, they were chosen the same for all isotopes and equal tothe amplitude of the tensor interaction was considered negligible. The selected parameters of the nn and np interactions are shown in Table 1.These values turned out to be close to the values obtained in [12], for heavy nuclei.They vary with the number of neutrons monotonously and slowly. In addition, $V_{pp} > V_{nn}$for all isotopes.This is due to the fact that the single-particle energies splinter for protons slightly more than for neutron holes. The table shows that the depth of the neutron-proton interaction is also somewhat greater for all isotopes and it slowly decreases with a decrease in the number of neutrons.

	U_W^n	U_S^n	U_W^{np}	U_S^{np}
^{100}Ru	-22	-16	-30	-24
^{102}Ru	-19	-14	-27	-22
^{104}Ru	-17	-11	-24	-19
^{106}Ru	-15	-10	-22	-17
^{108}Ru	-12	-8	-20	-14

They computed the entire low-lying spectrum of even ruthenium isotopes with N = 58, 60, 62, 64, 66. The obtained energy values of the levels of these nuclei are compared with their experimental data, which are listed in Table 2.

Ядра	^{100}Ru		^{102}Ru		^{104}Ru		^{106}Ru		^{108}Ru	
	Exp.	Theor.	Exp.	Theor.	Exp.	Theor.	Exp.	Theor.	Exp.	Theor.
0_1^+	0	0	0	0	0	0	0	0	0	0
2_1^+	0,54	0,52	0,48	0,46	0,36	0,35	0,27	0,25	0,24	0,23
4_1^+	1,23	1,18	1,11	1,09	0,89	0,81	0,71	0,66	0,67	0,61
6_1^+	2,08	1,97	1,87	1,72	1,56	1,45	1,30	1,19	1,22	1,11
8_1^+	3,06	2,80	2,70	2,51	2,32	2,13	1,97	1,76	-	1,61
10_1^+	4,09	3,56	3,43	3,15	3,11	2,91	-	2,46	-	2,13
0_2^+	1,13	1,05	0,94	1,01	0,99	0,90	0,99	0,91	1,09	1,05
2_3^+	1,87	1,69	1,58	1,45	1,52	1,37	1,39	1,29	1,25	1,12
2_2	1,36	1,22	1,10	1,02	0,89	0,75	0,79	0,81	0,71	0,80
3_1	1,88	1,65	1,52	1,41	1,24	1,04	1,09	0,94	0,97	0,88
4_2	-	1,24	1,80	1,34	1,50	1,35	-	1,29	1,18	1,01
5_1	-	1,89	2,22	1,96	1,87	1,59	-	1,32	1,49	1,18

The choice of these nuclei for research is primarily due to the fact that their low-energy states manifest themselves most purely in the presentation of generalized seniority, as noted in [4, 12]. It should be noted that there is a good agreement between the calculated values of E and the experiment for states with small angular valuesJ,

Table 3 - Relationships of E2 transitions between states in Ru nuclei.

$\frac{J_i \to J_f}{J_i' \to J_f'}$	^{100}Ru		^{102}Ru		^{104}Ru		^{106}Ru		^{108}Ru	
	Exp.	Theor.	Exp.	Theor.	Exp.	Theor.	Exp.	Theor.	Exp.	Theor.
$\frac{4_1 \to 2_1}{2_1 \to 0_1}$	1,50 ±0,21	1,47	1,6 ±0,3	1,5	1,43 ±0,20	1,44	-	1,75	-	1,83
$\frac{0_2 \to 2_1}{2_1 \to 0_1}$	0,99 ±0,20	0,95	0,76 ±0,15	0,71	0,46 ±0,06	0,41	-		-	
$\frac{2_2^+ \to 2_1}{2_1 \to 0_1}$	0,91 ±0,18	0,83	0,90 ±0,15	0,84	1,0 ±0,02	0,82	11,6 ±1,2	5,2	10,2 ±1,0	6,1
$\frac{2_2 \to 2_1}{2_2 \to 0_1}$	15,4 ±0,5	12,0	-	11,4	25 ±4,0	14,5	-	16,0	-	22,4
$\frac{2_3 \to 4_1}{2_3 \to 2_1}$	16,2 ±2,5	17,4	8,0 ±2,0	12,6	-	15,2		18,3		18,9
$\frac{3_1 \to 2_2}{3_1 \to 2_1}$	11 ±5,6	13,2	27 ±3,0	29,3	26 ±3,0	10,4	26 ±10	22,4	17 ±3	25,6

for which the main role is played by the interaction potentials of like nucleonsV_{pp},V_{nn}.The usefulness of these potentials in these cases is determined by their properties, which well preserve the scheme of generalized seniority.At the same time, such a purity of the quantum number of the generalized seniorityϑ, greatly simplifies the calculation procedure and leads to close real energy values for small values of the spins J, as can be seen from Table 2,3,4.

Table 4 - Relationships of $\delta(E2/M1)$ transitions between states in Ru nuclei

$J_i - J_f$	^{100}Ru		^{102}Ru		^{104}Ru		^{106}Ru		^{108}Ru	
	Exp.	Theor.	Exp.	Theor.	Exp.	Theor.	Exp.	Theor.	Exp.	Theor.
$2_2 \to 2_1$	3,4 ±0,8	2,9	-6,0 ±0,2	-2,7	-9,0	-4,4	7,1 ±1,4	5,6	4,3 ±0,8	2,9
$2_3 \to 2_1$		3,6	0,25 ±0,03	1,2	0,43 ±0,11	0,25	0,24 ±0,12	0,43	0,87 ±0,56	0,56
$3_1 \to 2_1$	0,45 ±0,1	0,9	0,90 ±0,15	0,6	-3,2 ±0,4	-1,7	-3,8 ±1,4	-5,4	-3,0 ±0,95	-1,4
$4_2 \to 4_1$	-	0,17	-	0,26	0,11 ±0,11	0,18	-	0,22	-	22,4
$3_1 \to 2_1$	-	-4,3	-7,2 (10)	-3,2	-	-	-	-2,3	-	0,27
$4_1 \to 2_1$	-	-	0,01 (5)	-	-	-	-	-	-	3,1
$5_1 \to 4_1$	-	-	-1,05 ±0,05	-	-	-	-	-	-	

However, in neutron-protoninteractionsV_{np}-begin to dominate the quadrupole-quadrupole interaction, which strongly mixes states with different senorities.If we restrict ourselves to several protons or neutrons in the valence shell, then the mixing of components in states with high senority, mainly the lower excited states, can be included in the calculations of the lower approximation of matrix diagonalization.Due to the quadrupole nature of the n-p forces, the components of the wave functions of such as$S_n^N S_p^N (D_n D_p)^0$fractions will mix states of different senoriths, which strongly wag the collectivity of D-pair-fermion states. Because of this, the discrepancy between the calculated and experimental values of the energy of states with large J becomes stronger, as can be seen from Table 2,3,4.Apparently, the hexadecapole component in pn-forces will play a certain role here.In addition, in such calculations, the single-particle energies were taken constant for all isotopes of the nucleus, whereas they can be varied by changing the number of nucleons in the shells.However, as shown by calculations for deformed isotopes$^{106}Ru$the main properties of the states in terms of energy are reproduced quite satisfactorily, since they change smoothly.

Table 3 and 4 shows the probability ratios of E2 transitions between different states of Ru isotopes. They also show satisfactory agreement of their calculated values with experimentally measured functions, especially for low-lying states.For levels with higher energies, these values differ significantly more. This is also mainly due to the strong mixing of states with different senorities due to quadrupole n-p interaction, as well as the neglect of the contributions of G-pair configurations to computational procedures.In many microscopic models with the method of mapping the fermion space into the bosonic one, the mapping methods are carried out not through the operators of fermion-boson transformations, but through the equalities of the matrix elements of the states in two spaces.As was noted [5, 6], these two spaces are interconnected through the bosonic and fermionic seniority. Such a connection is especially important for states with high senority, in which np-forces between particles play the main role. And they can give a fairly large contribution and bosons with large orbital moments.This leads to a change in the energies of the d-boson states. Therefore, in such calculations with space, it was necessary to renormalize the parameters of bosons included in the theory or parameters of quadrupole interactions in microscopic calculations. In our calculations, it was not necessary to lead such processes of renormalization of the parameters of theories. Despite this, the fermion theory with generalized senority, in general, gives a satisfactory smooth description of the properties of nuclei with average atomic weights.

CONCLUSION

On the basis of the generalized quasispin approach, the microscopic structure of the collective states of even ruthenium isotopes in the low-energy region was studied. To solve the many-particle problem in the space of generalized quasispins of, the potentials of nn, pp, npinteractions are taken in the most general form, the parameters of which are chosen from a comparison of the calculated values with their experimental values.

The description of microstates of nuclei with the help of generalized senority and double tensors, which express pairwise interaction of nucleons, greatly simplifies the procedure for calculating matrix elements, which give a good confirmation of the experimental facts on energy of states and on probabilities of electromagnetic transitions between them, especially for states with small quantum numbers J. However, the quadrupole interaction operator between different nucleons V_{pn}-strongly mixes states with different senorities.

This fact strongly influences the formation of the collectivity of the D-fermion states in the systems, which weakens the consistency of the calculated and experimental values of the energy levels, as well as the relative probabilities of E2 transitions between them, with large spins J.Nevertheless, the main properties of deformed nuclei are transferred quite satisfactorily. In some microscopic calculations using the method of mapping the fermion space into the bosonic space due to the connection of these representations through fermionic and bosonicsenority, the contributions of pairwise states with large orbital moments, for example, G-states, increase.Apparently, the divergence of our calculations for a state with large angular moments and with higher excitation energies also requires taking into account the contributions to them of G-pair formations. In addition, in the composition of high-energy levels, the role of that part of the complete Hamiltonian that was not included in the generalized seignity scheme is important.

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ЯДРОЛАР КОЛЛЕКТИВТІК КҮЙЛЕРІНІҢ ФЕРМИОНДЫҚ ТЕОРИЯЛАРЫН ЖӘНЕ ОНЫ НАҚТЫ ЖҮЙЕГЕ ҚОЛДАНУ

Аннотация. Атомдық салмағы орташа ядролар коллективтік күйлерінің микроскоптық құрылымы қосарланған нуклондық қабықша моделінде зерттеледі. Теорияда үлкен фермиондық кеңістік жалпыланған сеньорити әдісі көмегімен реалды SD-операторлар шеңберіне дейін қысқара кесілді және бірбөлшектік деңгейлер жіктелуінің жүйедегі күйлерін коллективтік құрылысына әсері де есепке алынды. Осындай көпбөлшектік мәселені шешу үшін жалпыланған квазиспин әдісі мен қосарлы тензорлар ұғымы қолданылды. Олар қосарлы потенциалдардың матрицаларын есептеуге өте қолайлы. Толық гамильтониан таза фермиондық кеңістікте дәл диагональдауы бірге енгізілді. Мұнда фермиондық операторларды бозондық түрге айналдырудың қажеті болмады. Әсерлесуші бозондар моделінің параметрлері фермиондық жолмен есептелді. Құрылған теория рутений ядросының N=58-66 жұпты изотоптары құрылысына қолданылды. Олардағы төмегі энергиялы деңгей спектрі және электромагниттік Е2-ауысу ықтималдығы есептеліп, алныған шамалар экспериментте табылған мәндерімен салыстырылды.

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ФЕРМИОННАЯ ТЕОРИЯ КОЛЛЕКТИВНЫХ СОСТОЯНИЙ ЯДЕР ЕЕ ПРИЛОЖЕНИЕ К СТРУКТУРЕ РЕАЛЬНЫХ СИСТЕМ

Аннотация. На основе нуклонно-парной оболочечной модели, в которой методом обобщенной сеньорити обрезается фермионное пространство ″реалистическими″ SD-операторами, изучены микроскопическая структура коллективных состояний ядер среднего атомного веса. При этом учтены влияния расщепления одночастичных уровней на коллективно-парную структуру системы. Для решения такой многочастичной задачи используется метод обобщенного квазиспина и двойные тензоры, облегчающие вычисления матричных элементов парных взаимодействий нуклонов. Полный гамильтониан диагонализуется точно в фермионном пространстве без применения процедуры отображения фермионных

операторов в бозонные. Вычислены параметры модели взаимодействующих бозонов на основе перелагаемого фермионного подхода. Теория приложена к изучению свойств коллективных состояний четных изотопов рутения с N=58-66. Вычислены спектр состояний низких энергии также вероятности E2-переходов в них и они сравнены с имеющимися экспериментальными данными.

REFERENCES

[1] Arima A., Iachello F. Interacting boson model of collective states. I-Ann. Phys. 1978, v.99, p. 253-317.
[2] Arima A., Iachello F. Interacting boson model of collective states. II-Ann. Phys. 1978, v 111, p. 201-238.
[3] Arima A., Iachello F. Interacting boson model of collective states. III-Ann. Phys. 1978, v 123, p. 468-492.
[4] Бактыбаев К. Описание коллективных возбудений ядер в модели взаимодействующих бозонов. ЯФ., 1979, Т.30, № 4 стр. 963-973.
[5] Talmi I. Generalized seniority and struture of semi-magic nuclei.=Nucl. Phys. 1971, v. A172, p.1;
[6] Lorazo B. - Generalized seniority and broken-pair coupling Scheme. Nucl. Phys., 1970, v. A 153, p. 255.
[7] Бактыбаев К., Абельдина Ж.К. - формализм обобщенного квазиспина в теории ядра, Изв. АН СССР, сер-физ., 1979, Т.43, № 11, стр. 229.
[8] Y-A Luo, F.Pan.Ch.Bahri, J.P.Drayer-Phys.Rev.C 71044304(2005).
[9] K.Baktybaev, A. Dalelkhankyzy, M.K.Baktybaev //SD- steam theory of collective excitations and the validity of the model of Interacting bosons // Adv. studies theor. Phys., 2012 vol.6, №28, p. 1399-1404.
[10] K.Baktybaev, A. Dalelkhankyzy, M.K.Baktybaev, N.Koilyk «The nucleon-pair shell description of collective excitations of spherical nuclei». Adv. Studies Theor. Phys., Vol. 7, 2013, no. 12.
[11] K.Baktybayev, A.Dalelkhankyzy and N.koilyk-Advanced Studies in Theoretical Physics, Vol.9, 2015,no.10,483-493.
[12] Ohtsuka T., Arima A., Iachello F. Nuclear shell-model and interacting bosons. NuclearPhys. 1978, v. A 309, n. 1-2, p. 1-33.
[13] Ohtsuka T., Arima A., Iachello F. Shell-model descrition interacting bosons. NuclearPhys. 1978, v. 76B, n.2, p. 139-143.
[14] Слив Л.А. Новые данные по исследованию околомагических ядер, Изв. АН СССР, 1972, сер-физ. Т.36 с. 2026
[15] Ichimura M., Arima A.-Quasi-Spin formalism and Matrix elements in the shall. Model. Pragr.Theor.Phys., v.36, №2, 1996, p.296-312.
[16] Wu C-L, Feng D.H., of all, //Nucl.Phys. 1994. v.A 570. P. 337.
[17] Yoshinaga N., Higashiyama K. // Phys. Rev. C 69, 054309 (2004).
Iachello F., Talmi I. Shell model foundations of the interacting boson model // Rev. Mod. Phy. V. 59, 1987. P. 339.
[18] Luo Y-A., Chen J-Q. // Phys. Rev. C 58, 589 (1998).
[19] Luo Y-A., Chen J-Q., Draayer J.P. Nucleon-pair Shell model calculations of the even-even Xe and Ba nuclei. Nucl. Phys. A 669 (2000), P. 101-118.
[20] Luo Y.A., Pan F., Bahri Ch., Dayer I.P. $S, D-$ pair shell model and the interacting boson model // Phys.Rev. C. 71. 044304. 2005.
[21] Takada K., Tazaki S., Yosumoto S. Dyson boson mapping and shell-model calculations for even-even nuclei // Progr. Theor. Phys. -2006. No. 1, -V 16. -P. 107.
[22] Бактыбаев К., Абельдина Ж.К. Структура состояний изотопов стронция с A=89-94 в оболочечной модели с обобщенной сеньорити, Изв. АН СССР, сер.физ. 1978, т. 42, стр. 116.
[23] Бегжанов Р.Б. и др. Справочник по ядерной физике, 1989, книга 1.
[24] Lederer C.M., Shirley V.S. Table of isotopes 7th ed. N.Y.1978.
[25] Mamyrbayev O. Zh., Shayakhmetova, A. S.); Seisenbekova P. B. The methodology of creating an intellectual environment of increasing the competence of students based on a bayesian approach//News of the National academy of sciences of the Republic of Kazakhstan. Series physico-mathematical. 2019. №4 (326). P. 50-58. https://doi.org/10.32014/2019.2518-1726.43

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INVERSE PROBLEM OF STURM-LIOUVILLE OPERATOR WITH NON-SEPARATED BOUNDARY VALUE CONDITIONS AND SYMMETRIC POTENTIAL

Abstract: In this paper, we prove uniqueness theorem, by one spectrum, for a Sturm-Liouville operator with non-separated boundary value conditions and a real continuous and symmetric potential. The research method differs from all previously known methods and is based on internal symmetry of the operator generated by invariant subspaces.
Keywords: Sturm-Liouville operator, spectrum, inverse Sturm-Liouville problem, Borg theorem, Ambartsumyan theorem, Levinson theorem, non-separated boundary value conditions, symmetric potential, invariant subspaces, differential operators, inverse spectral problems.

1. Introduction

We study the following inverse spectral problem for the Sturm-Liouville operator:

$$Ly := y'' + q(x)y,\ x \in (0, 1),$$

on a finite interval (0, 1) with non-separated boundary value conditions. Inverse problems consist in restoring the coefficients of differential operators by their spectral characteristics. Such problems often arise in mathematics and its applications.

Inverse problems for differential operators with decaying boundary value conditions have been thoroughly studied (see monographs [1–5] and references). More difficult inverse problems for Sturm – Liouville operators with non-decaying boundary value conditions were studied in [6–17] and other works. In particular, periodic boundary-value problem was considered in [6, 7, 9, 14]. I. V. Stankevich [6] proposed formulation of the inverse problem and proved the corresponding uniqueness theorem. V. A. Marchenko and I. V. Ostrovsky [7] characterized spectrum of a periodic boundary-value problem in terms of a special conformal mapping. The conditions proposed in [7] are difficult to verify. Another method, used in [9], made it possible to obtain necessary and sufficient conditions for solvability of the inverse problem in the periodic case that are more convenient to verify. Similar results were obtained in [9], and for another type of boundary conditions, namely

$$y'(0) - ay(0) + by(\pi) = y'(\pi) + dy(\pi) - by(0) = 0.$$

Later similar results were obtained in [12, 13]. In the paper [18], the case when the potential q is symmetric with respect to the middle of interval, i.e., $q(x) = q(\pi - x)$ a.e. on $(0, \pi)$, was studied, and for this case a solution of the inverse spectral problem was constructed and a spectrum was given. The symmetric case requires nontrivial changes in the method and allows us to specify less spectral information than in the general case. Some results for the symmetric case were obtained in [10] and [17] - [24].

The inverse problems of spectral analysis are understood as problems of reconstructing a linear operator from one or another of its spectral characteristics. The first significant result in this direction was obtained in 1929 by V.A. Ambardzumyan [25]. He proved the following theorem.

By $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ we denote eigenvalues of the following Sturm-Liouville problem

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$y'(0) = 0, \ y'(\pi) = 0; \tag{1.2}$$

where $q(x)$ is a real continuous function. If

$$\lambda_n = n^2 \ (n = 0,1,2,\dots) \ \text{то} \ q(x) \equiv 0.$$

The first mathematician who drew attention to importance of this Ambardzumyan result was the Swedish mathematician Borg. He performed the first systematic research of one of important inverse problems, namely, the inverse problem for the classical Sturm – Liouville operator of the form (1.1) by the spectra [26]. Borg showed that in the general case one spectrum of the Sturm - Liouville operator does not determine it, so the Ambartsumyan result is an exception to the general rule. In the same paper [26], Borg showed that two spectra of the Sturm – Liouville operator (under various boundary conditions) uniquely determine it. More precisely, Borg proved the following theorem.

Borg Theorem.

Let the equations

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$
$$-z'' + p(x)z = \lambda z, \tag{1.3}$$

have the same spectrum under the boundary value conditions

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma y(\pi) + \delta y'(\pi) = 0; \end{cases} \tag{1.4}$$

under the boundary value conditions

$$\begin{cases} \alpha y(0) + \beta y'(0) = 0, \\ \gamma' y(\pi) + \delta' y'(\pi) = 0. \end{cases} \tag{1.4$'$}$$

Then $q(x) = p(x)$ almost everywhere on the segment $[0, \pi]$, if

$$\delta \cdot \delta' = 0, \qquad |\delta| + |\delta'| > 0.$$

Soon after the Borg work, important studies on the theory of inverse problems were carried out by Levinson [27], in particular, he proved that if $q(\pi - x) = q(x)$, then the Sturm – Liouville operator

$$-y'' + q(x)y = \lambda y, \tag{1.1}$$

$$\begin{cases} y'(0) - hy(0) = 0, \\ y'(\pi) + hy(\pi) = 0 \end{cases} \tag{1.5}$$

is reconstructed by one spectrum.

A number of B.M. Levitan works [28, 29] are devoted to reconstruction of the Sturm – Liouville operator by one and two spectra.

This work is devoted to a generalization of the theorems of Ambartsumian [25] and Levinson [27], in particular, our results contain the results of these authors. Research method of this work appeared under influence of [30] - [32], and differs from all previously known methods.

1. Research Method.

Idea of this work is very simple. Having studied in detail contents of [1, 3], we realized that both of these operators have an invariant subspace. If for the linear operator L, we have the formulas

$$LP = PL^*, \qquad QL = L^*Q,$$

where P, Q are orthogonal projectors, satisfying the condition $P + Q = I$, then the operators L and L^* have invariant subspaces, sometimes restriction of these operators to these invariant subspaces, under certain conditions, form a Borg pair.

2. Research Results.

In the Hilbert space $H = L^2(0,\pi)$ we consider the Sturm – Liouville operator.

$$Ly = -y'' + q(x)y,\ x \in (0,\pi); \tag{3.1}$$

$$\begin{cases} a_{11}y(0) + a_{12}y'(0) + a_{13}y(\pi) + a_{14}y'(\pi) = 0, \\ a_{21}y(0) + a_{22}y'(0) + a_{23}y(\pi) + a_{24}y'(\pi) = 0 \end{cases} \tag{3.2}$$

where $q(x)$ is a continuous complex function, a_{ij} $(i = 1,2;\ j = 1,2,3,4)$ are arbitrary complex coefficients, and by Δ_{ij} $(i = 1,2;\ j = 1,2,3,4)$ we denote minors of the boundary matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

Suppose that $\Delta_{13} \neq 0$, then the Sturm – Liouville operator (3.1) – (3.2) has the following form

$$Ly = -y'' + q(x)y,\ x \in (0,\pi); \tag{3.1}$$

$$\begin{cases} \Delta_{13}y(0) - \Delta_{32}y'(0) - \Delta_{34}y'(\pi) = 0, \\ \Delta_{12}y'(0) + \Delta_{13}y(\pi) + \Delta_{14}y'(\pi) = 0, \end{cases} \tag{3.3}$$

and its conjugate operator L^+ has the form

$$L^+z = -z'' + \overline{q(x)}z,\ x \in (0,\pi); \tag{3.1$^+$}$$

$$\begin{cases} \overline{\Delta_{13}}z(0) - \overline{\Delta_{32}}z'(0) - \overline{\Delta_{12}}z'(\pi) = 0, \\ \overline{\Delta_{34}}z'(0) + \overline{\Delta_{13}}z(\pi) + \overline{\Delta_{14}}z'(\pi) = 0. \end{cases} \tag{3.3$^+$}$$

Let P and Q be orthogonal projectors, defined by the formulas

$$Pu(x) = \frac{u(x)+u(\pi-x)}{2},\ Qv(x) = \frac{v(x)-v(\pi-x)}{2} \tag{3.4}$$

The main result of this paper is the following theorem.

Theorem 3.1. If $\Delta_{13} \neq 0$, and

1) $LP = PL^+$; (3.5)
2) $QL = L^+Q$; (3.6)
3) $\Delta_{12} = -\Delta_{34}$; (3.7)

then the Sturm – Liouville operator (3.1) – (3.3) is reconstructed by one spectrum.

3. Discussion.

In this section we prove the theorem and discuss the obtained results. The following Lemmas 4.1 and 4.2 can have independent values.

Lemma 4.1. If for a linear and discrete operator L, the following equalities hold:

1) $LP = PL^+$; (3.5)
2) $QL = L^+Q$; (3.6)
3) $P + Q = I$; (3.8)

where P, Q are orthogonal projectors, and I is unit operator, then all its eigenvalues are real.

Proof.

Let $LP = PL^*$, $QL = L^*Q$; then

$$(LP)^* = P^*L^* = PL^* = LP;$$

$$(QL)^* = L^*Q^* = L^*Q = QL;$$

i.e. operators LP and QL are selfadjoint, therefore their eigenvalues are real.

If $Ly = \lambda y,\ y \neq 0,$ then $QLy = \lambda Qy,\ L^+Qy = \lambda Qy,\ L^+Q(Qy) = \lambda Qy,\ QL(Qy) = \lambda Qy$ if $Qy \neq 0$, then λ is a real quantity; if $Qy = 0$, then $y = Py \neq 0$, and $LPy = \lambda Py,\ LP(Py) = \lambda Py$. Consequently, λ is again real quantity.

The following lemma shows that the spectrum $\sigma(L)$ of the operator L splits into two parts; therefore, the operator L, apparently, also splits into two parts. Furthermore, we will see that this is exactly what happens, and more precisely, these parts form a Borg pair under a certain condition.

Lemma 4.2. If L is a linear discrete operator, satisfying the conditions:

1) $LP = PL^+$; (3.5)
2) $QL = L^+Q$; (3.6)
3) $P + Q = I$; (3.8)

where P, Q are orthogonal projectors, and I is identity operator, then we have

$$\sigma(L) = \sigma(L_1) \cup \sigma(L_2). \quad (3.9)$$

where $L_1 = LP,\ L_2 = QL,\ \sigma(L)$ is a spectrum of the operator L.

Proof.

If $Ly = \lambda y,\ y \neq 0$, then $QLy = \lambda Qy,\ L^+Qy = \lambda Qy,\ L^+Q(Qy) = \lambda Qy,\ L_2Qy = \lambda Qy$. If $Qy \neq 0$, then $\lambda \in \sigma(L_2)$. If $Qy = 0$, then $y = Py \neq 0$ and $LPy = \lambda Py,\ LP(Py) = \lambda Py,\ L_1Py = \lambda Py$. Consequently, $\lambda \in \sigma(L_1)$.

Hence, $\sigma(L) \subset \sigma(L_1) \cup \sigma(L_2)$.

If $\lambda \neq 0$, and $\lambda \in \sigma(L_1) \cup \sigma(L_2)$, then

a) If $\lambda \in \sigma(L_1)$, then $\exists u \neq 0$, such that $u \in H_1,\ L_1u = \lambda u,\ LPu = \lambda u, \rightarrow Lu = \lambda u$. Consequently, $\lambda \in \sigma(L)$.

b) If $\lambda \in \sigma(L_2)$, then $\exists v \in H_2,\ v \neq 0$ such that $L_2v = \lambda v,\ QLv = \lambda v,\ L^+Qv = \lambda v,\ L^+v = \lambda v$. Thus, $\lambda \in \sigma(L^+) = \sigma(L)$.

c) If $0 \in \sigma(L_1) \cup \sigma(L_2)$, then if $0 \in \sigma(L_1)$, then $L_1u = 0,\ u \in H_1,\ LPu = 0, => Lu = 0, => 0 \in \sigma(L)$. If $0 \in \sigma(L_2)$, then $L_2v = 0,\ v \in H_2,\ QLv = 0, =>\ L^+Qv = 0, L^+v = 0, => 0 \in \sigma(L^+) = \sigma(L)$.

The following two Lemmas 4.3 and 4.4 refine boundary conditions of the Sturm - Liouville operators with invariant subspaces.

Lemma 4.3. If

a) $\Delta_{13} \neq 0$;

b) $LP = PL^+$;

then the following formulas hold

1) $\Delta_{12} + \Delta_{32} = \Delta_{14} + \Delta_{34}$;

2) $\frac{\Delta_{12}-\Delta_{14}}{\Delta_{13}} = \overline{\left(\frac{\Delta_{12}-\Delta_{14}}{\Delta_{13}}\right)} = \frac{\Delta_{34}-\Delta_{32}}{\Delta_{13}}$;

3) $\overline{q(x)} = q(x),\ q(\pi - x) = q(x)$;

and the operators L and L^+ have the following forms:

a) $Ly = -y'' + q(x)y,\ x \in (0, \pi)$;

$$\begin{cases} y(0) - y(\pi) - \dfrac{\Delta_{12} + \Delta_{32}}{\Delta_{13}}[y'(0) + y'(\pi)] = 0, \\ \Delta_{12}y'(0) + \Delta_{13}y(\pi) + \Delta_{14}y'(\pi) = 0. \end{cases}$$

b) $L^+z = -z'' + q(x)z,\ x \in (0, \pi)$;

$$\begin{cases} z(0) + z(\pi) + \dfrac{\overline{\Delta_{12}} - \overline{\Delta_{14}}}{\overline{\Delta_{13}}}[z'(0) + z'(\pi)] = 0, \\ \overline{\Delta_{13}}z(0) - \overline{\Delta_{32}}z'(0) - \overline{\Delta_{12}}z'(\pi) = 0. \end{cases}$$

Proof.

Assume that

$$LP = PL^+; \quad (3.5)$$

From the condition $z \in D(L^{+})$ it follows that $y = Pz \in D(L)$, therefore we have the following equalities:

$$\begin{cases} \Delta_{13}\dfrac{z(0)+z(\pi)}{2} - \Delta_{32}\dfrac{z'(0)-z'(\pi)}{2} - \Delta_{34}\dfrac{z'(\pi)-z'(0)}{2} = 0, \\ \Delta_{12}\dfrac{z'(0)-z'(\pi)}{2} + \Delta_{13}\dfrac{z(\pi)+z(0)}{2} + \Delta_{14}\dfrac{z'(\pi)-z'(0)}{2} = 0. \end{cases}$$

$$\begin{cases} \Delta_{13}\dfrac{z(0)+z(\pi)}{2} + (\Delta_{34}-\Delta_{32})\dfrac{z'(0)-z'(\pi)}{2} = 0, \\ \Delta_{13}\dfrac{z(0)+z(\pi)}{2} + (\Delta_{12}-\Delta_{14})\dfrac{z'(0)-z'(\pi)}{2} = 0. \end{cases}$$

From (3.5) it follows that $\Delta_{12} + \Delta_{32} = \Delta_{14} + \Delta_{34}$, then $\Delta_{34} - \Delta_{32} = \Delta_{12} - \Delta_{14}$, and two boundary conditions merge into one boundary condition. Hence,

$$\Delta_{13}\frac{z(0)+z(\pi)}{2} + (\Delta_{12}-\Delta_{14})\frac{z'(0)-z'(\pi)}{2} = 0. \quad (4.1)$$

Summing up the boundary conditions (3.3)$^{+}$, we get

$$\overline{\Delta_{13}}[z(0)+z(\pi)] + (\overline{\Delta_{34}} - \overline{\Delta_{32}})z'(0) + (\overline{\Delta_{14}} - \overline{\Delta_{12}})z'(\pi) = 0,$$

$$\overline{\Delta_{13}}[z(0)+z(\pi)] + (\overline{\Delta_{12}} - \overline{\Delta_{14}})z'(0) - (\overline{\Delta_{12}} - \overline{\Delta_{14}})z'(\pi) = 0,$$

$$\overline{\Delta_{13}}[z(0)+z(\pi)] + (\overline{\Delta_{12}} - \overline{\Delta_{14}})[z'(0)-z'(\pi)] = 0. \quad (4.2)$$

From (4.1) and (4.2) we write the system of equations:

$$\Delta_{13}\frac{[z(0)+z(\pi)]}{2} + (\Delta_{12}-\Delta_{14})\frac{[z'(0)-z'(\pi)]}{2} = 0,$$

$$\overline{\Delta_{13}}\frac{[z(0)+z(\pi)]}{2} + (\overline{\Delta_{12}}-\overline{\Delta_{14}})\frac{[z'(0)-z'(\pi)]}{2} = 0;$$

This system has a nontrivial solution, therefore,

$$\begin{vmatrix} \Delta_{13} & \Delta_{12}-\Delta_{14} \\ \overline{\Delta_{13}} & \overline{\Delta_{12}}-\overline{\Delta_{14}} \end{vmatrix} = 0 \text{ или } \frac{\Delta_{12}-\Delta_{14}}{\Delta_{13}} = \overline{\left(\frac{\Delta_{12}-\Delta_{14}}{\Delta_{13}}\right)}.$$

Further, subtracting the second boundary condition from the first condition (see 3.3), we obtain

$$\Delta_{13}[y(0)-y(\pi)] - (\Delta_{12}+\Delta_{32})y'(0) - (\Delta_{34}+\Delta_{14})y'(\pi) = 0,$$

$$\Delta_{13}[y(0)-y(\pi)] - (\Delta_{12}+\Delta_{32})[y'(0)+y'(\pi)] = 0,$$

$$y(0)-y(\pi) - \frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}[y'(0)+y'(\pi)] = 0$$

Now we study properties of the differential expression L. From the formula $LP = PL^{+}$, we get

$$LPz = L^{\circ}\frac{z(x)+z(\pi-x)}{2} = -\frac{z''(x)+z''(\pi-x)}{2} + q(x)\frac{z(x)+z(\pi-x)}{2};$$

$$PL^{+}z = P^{\circ}\left[-z'' + \overline{q(x)}z\right] = -\frac{z''(x)+z''(\pi-x)}{2} +$$

$$+\frac{\bar{q}(x)z(x)+\bar{q}(\pi-x)z(\pi-x)}{2};$$

$$q(x)z(x) - q(x)z(\pi-x) = \bar{q}(x)z(x) + \bar{q}(\pi-x)z(\pi-x),$$

$$\begin{cases} [q(x)-\bar{q}(x)]z(x)+[q(x)-\bar{q}(\pi-x)]z(\pi-x)=0, \\ [q(\pi-x)-\bar{q}(\pi-x)]z(\pi-x)+[q(\pi-x)-\bar{q}(x)]z(x)=0; \end{cases} \quad (4.3)$$

$$\Delta=\begin{vmatrix} q(x)-\bar{q}(x) & q(x)-\bar{q}(\pi-x) \\ q(\pi-x)-\bar{q}(x) & q(\pi-x)-\bar{q}(\pi-x) \end{vmatrix}=0;$$

$$[q(x)-\bar{q}(x)][q(\pi-x)-\bar{q}(\pi-x)]-$$

$$[q(x)-\bar{q}(\pi-x)][q(\pi-x)-\bar{q}(x)]=0;$$

$$q(x)q(\pi-x)-q(x)\bar{q}(\pi-x)-\bar{q}(x)q(\pi-x)+\bar{q}(x)\bar{q}(\pi-x)=$$

$$=q(x)q(\pi-x)-q(x)\bar{q}(x)-\bar{q}(\pi-x)q(\pi-x)+\bar{q}(\pi-x)\bar{q}(x);$$

$$q(x)\bar{q}(\pi-x)+\bar{q}(x)q(\pi-x)=q(x)\bar{q}(x)+\bar{q}(\pi-x)q(\pi-x),$$

$$q(x)[\bar{q}(\pi-x)-\bar{q}(x)]+q(\pi-x)[\bar{q}(x)-\bar{q}(\pi-x)]=0,$$

$$[\bar{q}(x)-\bar{q}(\pi-x)]\cdot[q(\pi-x)-q(x)]=0,$$

$$|q(x)-q(\pi-x)|^2=0, \implies q(x)=q(\pi-x).$$

Further, from (4.3) we get

$$[q(x)-\bar{q}(x)]z(x)+[q(x)-\bar{q}(x)]z(\pi-x)=0,$$

$$[q(x)-\bar{q}(x)][z(x)+z(\pi-x)]=0, \implies q(x)-\bar{q}(x)=0.$$

Lemma 4.4. If

a) $\Delta_{13}\neq 0$;

b) $QL=L^{+}Q$,

then

1) $\overline{\Delta_{12}+\Delta_{32}}=\Delta_{14}+\Delta_{34}$;

2) $\overline{\left(\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}\right)}=\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}=\frac{\Delta_{14}+\Delta_{34}}{\Delta_{13}}$;

3) $q(\pi-x)=q(x), \bar{q}(x)=q(x)$,

and the operators L and L^{+} have the form

4) $Ly=-y''+q(x)y,\ x\in(0,\pi)$;

$$\begin{cases} y(0)-y(\pi)-\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}[y'(0)+y'(\pi)]=0, \\ \Delta_{12}y'(0)+\Delta_{13}y(\pi)+\Delta_{14}y'(\pi)=0. \end{cases}$$

5) $L^{+}z=-z''+\bar{q}(x)z,\ x\in(0,\pi)$;

$$\begin{cases} z(0)+z(\pi)+\frac{\overline{\Delta_{12}}-\overline{\Delta_{14}}}{\overline{\Delta_{13}}}[z'(0)-z'(\pi)]=0, \\ \overline{\Delta_{13}}z(0)-\overline{\Delta_{32}}z'(0)-\overline{\Delta_{12}}z'(\pi)=0. \end{cases}$$

Proof.

Suppose that the following equality holds:

$$QL=L^{+}Q$$

then the condition $y(x)\in D(L)$ implies that $z=Qy\in D(L^{+})$, therefore the following equalities hold:

$$z(x)=\frac{y(x)-y(\pi-x)}{2}, \qquad z'(x)=\frac{y'(x)+y'(\pi-x)}{2};$$

$$\begin{cases} \overline{\Delta_{13}}\frac{y(0)-y(\pi)}{2}-\overline{\Delta_{32}}\frac{y'(0)+y'(\pi)}{2}-\overline{\Delta_{12}}\frac{y'(\pi)+y'(0)}{2}=0, \\ \overline{\Delta_{34}}\frac{y'(0)+y'(\pi)}{2}+\overline{\Delta_{13}}\frac{y(\pi)-y(0)}{2}+\overline{\Delta_{14}}\frac{y'(\pi)+y'(0)}{2}=0; \end{cases}$$

$$\begin{cases} \overline{\Delta_{13}}\dfrac{y(0)-y(\pi)}{2}-(\overline{\Delta_{12}}+\overline{\Delta_{32}})\dfrac{y'(0)+y'(\pi)}{2}=0, \\ -\overline{\Delta_{13}}\dfrac{y(0)-y(\pi)}{2}+(\overline{\Delta_{14}}+\overline{\Delta_{34}})\dfrac{y'(0)+y'(\pi)}{2}=0; \end{cases}$$

$$\begin{cases} \overline{\Delta_{13}}\dfrac{y(0)-y(\pi)}{2}-(\overline{\Delta_{12}}+\overline{\Delta_{32}})\dfrac{y'(0)+y'(\pi)}{2}=0, \\ \overline{\Delta_{13}}\dfrac{y(0)-y(\pi)}{2}-(\overline{\Delta_{14}}+\overline{\Delta_{34}})\dfrac{y'(0)+y'(\pi)}{2}=0. \end{cases}$$

From $QL=L^{+}Q$ it follows that $\Delta_{12}+\Delta_{32}=\Delta_{14}+\Delta_{34}$, therefore there is only one boundary condition

$$\overline{\Delta_{13}}\frac{y(0)-y(\pi)}{2}-(\overline{\Delta_{12}}+\overline{\Delta_{32}})\frac{y'(0)+y'(\pi)}{2}=0. \qquad (4.4)$$

Subtracting the second boundary condition from the first boundary condition in (3.3), we obtain

$$\Delta_{13}[y(0)-y(\pi)]-(\Delta_{12}+\Delta_{32})y'(0)-(\Delta_{14}+\Delta_{34})y'(\pi)=0,$$

$$\Delta_{13}\frac{[y(0)-y(\pi)]}{2}-(\Delta_{12}+\Delta_{32})\frac{[y'(0)+y'(\pi)]}{2}=0. \qquad (4.5)$$

Combining the boundary conditions (4.4) - (4.5), we have

$$\begin{cases} \overline{\Delta_{13}}\dfrac{y(0)-y(\pi)}{2}-(\overline{\Delta_{12}}+\overline{\Delta_{32}})\dfrac{y'(0)+y'(\pi)}{2}=0, \\ \Delta_{13}\dfrac{[y(0)-y(\pi)]}{2}-(\Delta_{12}+\Delta_{32})\dfrac{[y'(0)+y'(\pi)]}{2}=0. \end{cases}$$

This system of equations has a nontrivial solution, therefore

$$\Delta=\begin{vmatrix} \overline{\Delta_{13}} & -(\overline{\Delta_{12}}+\overline{\Delta_{32}}) \\ \Delta_{13} & -(\Delta_{12}+\Delta_{32}) \end{vmatrix}=0, \Rightarrow \overline{\left(\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}\right)}=\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}.$$

Сложив граничных условий $(3.3)^{+}$, имеем

$$\overline{\Delta_{13}}[z(0)+z(\pi)]+(\overline{\Delta_{34}}-\overline{\Delta_{32}})z'(0)+(\overline{\Delta_{14}}-\overline{\Delta_{12}})z'(\pi)=0,$$

$$\overline{\Delta_{13}}[z(0)+z(\pi)]+(\overline{\Delta_{12}}-\overline{\Delta_{14}})z'(0)-(\overline{\Delta_{12}}-\overline{\Delta_{14}})z'(\pi)=0,$$

$$\overline{\Delta_{13}}[z(0)+z(\pi)]+(\overline{\Delta_{12}}-\overline{\Delta_{14}})[z'(0)-z'(\pi)]=0.$$

Consequently, boundary conditions of the operators L and L^{+} have the following forms:

$$L: \begin{cases} y(0)-y(\pi)-\frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}[y'(0)+y'(\pi)]=0, \\ \Delta_{12}y'(0)+\Delta_{13}y(\pi)+\Delta_{14}y'(\pi)=0; \end{cases}$$

$$L^{+}: \begin{cases} z(0)+z(\pi)+\frac{\overline{\Delta_{12}}-\overline{\Delta_{14}}}{\overline{\Delta_{13}}}[z'(0)-z'(\pi)]=0, \\ \overline{\Delta_{13}}z(0)-\overline{\Delta_{32}}z'(0)-\overline{\Delta_{12}}z'(\pi)=0. \end{cases}$$

Further, from the formula $QL=L^{+}Q$, we get

$$QLy=Q^{\circ}[-y''+q(x)y]=-\frac{y''(x)-y''(\pi-x)}{2}+$$
$$+\frac{q(x)y(x)-q(\pi-x)y(\pi-x)}{2};$$

$$L^{+}Qy=L^{+}\left[\frac{y(x)-y(\pi-x)}{2}\right]=$$
$$=-\frac{y''(x)-y''(\pi-x)}{2}+\bar{q}(x)\frac{y(x)-y(\pi-x)}{2};$$

$$q(x)y(x) - q(\pi - x)y(\pi - x) = \bar{q}(x)y(x) - \bar{q}(x)y(\pi - x),$$
$$[q(x) - \bar{q}(x)]y(x) + [\bar{q}(x) - q(\pi - x)]y(\pi - x) = 0, \quad (4.6)$$
$$[q(\pi - x) - \bar{q}(\pi - x)]y(\pi - x) + [\bar{q}(\pi - x) - q(x)]y(x) = 0;$$
$$\Delta = \begin{vmatrix} q(x) - \bar{q}(x) & \bar{q}(x) - q(\pi - x) \\ \bar{q}(\pi - x) - q(x) & q(\pi - x) - \bar{q}(\pi - x) \end{vmatrix} = 0,$$
$$[q(x) - \bar{q}(x)] \cdot [q(\pi - x) - \bar{q}(\pi - x)] -$$
$$-[\bar{q}(x) - q(\pi - x)][\bar{q}(\pi - x) - q(x)] = 0,$$
$$q(x)q(\pi - x) - q(x)\bar{q}(\pi - x) - \bar{q}(x)q(\pi - x) + \bar{q}(x)\bar{q}(\pi - x) =$$
$$= \bar{q}(x)\bar{q}(\pi - x) - \bar{q}(x)q(x) - q(\pi - x)\bar{q}(\pi - x) + q(\pi - x)q(x),$$
$$q(x)\bar{q}(\pi - x) + \bar{q}(x)q(\pi - x) = \bar{q}(x)q(x) + q(\pi - x)\bar{q}(\pi - x),$$
$$q(x)[\bar{q}(\pi - x) - \bar{q}(x)] + q(\pi - x)[\bar{q}(x) - \bar{q}(\pi - x)] = 0,$$
$$[\bar{q}(x) - \bar{q}(\pi - x)][q(\pi - x) - q(x)] =$$
$$= |q(x) - q(\pi - x)|^2 = 0, \Rightarrow q(x) = q(\pi - x).$$

From (4.6) we have

$$[q(x) - \bar{q}(x)][y(x) - y(\pi - x)] = 0, \Rightarrow q(x) - \bar{q}(x) = 0.$$

The previous Lemmas 4.3 and 4.4 yield the following theorem.

Theorem 4.1. If

a) $\Delta_{13} \neq 0$;

b) $LP = PL^{+}$;

c) $QL = L^{+}Q$,

then

1) $\overline{\left(\frac{\Delta_{12}+\Delta_{32}}{\Delta_{24}}\right)} = \frac{\Delta_{12}+\Delta_{32}}{\Delta_{24}} = \frac{\Delta_{14}+\Delta_{34}}{\Delta_{24}}$;

2) $\overline{\left(\frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}}\right)} = \frac{\Delta_{14}-\Delta_{12}}{\Delta_{24}} = \frac{\Delta_{32}-\Delta_{34}}{\Delta_{24}}$;

3) $q(\pi - x) = q(x), \bar{q}(x) = q(x)$;

and the operators L and L^{+} have the forms

4) $$Ly = -y'' + q(x)y, \ x \in (0, \pi);$$
$$\begin{cases} y(0) - y(\pi) - \frac{\Delta_{12}+\Delta_{32}}{\Delta_{13}}[y'(0) + y'(\pi)] = 0, \\ \Delta_{12}y'(0) + \Delta_{13}y(\pi) + \Delta_{14}y'(\pi) = 0. \end{cases}$$

5) $$L^{+}z = -z'' + \bar{q}(x)z, \ x \in (0, \pi);$$
$$\begin{cases} z(0) + z(\pi) + \frac{\overline{\Delta_{12}} - \overline{\Delta_{14}}}{\overline{\Delta_{13}}}[z'(0) - z'(\pi)] = 0, \\ \overline{\Delta_{13}}z(0) - \overline{\Delta_{32}}z'(0) - \overline{\Delta_{12}}z'(\pi) = 0. \end{cases}$$

Further from the formulas $LP = PL^{+}$ we note that the operator $L_1 = LP$ acts in the subspace $H_1 = PH$, where $H = L^2(0, \pi)$. Assuming

$$u(x) = Py(x) = \frac{y(x) + y(\pi - x)}{2},$$

we have

$$u'(x) = \frac{y'(x) - y'(\pi - x)}{2}.$$

Then Theorem 4.1 implies that

$$L_1 u = -u'' + q(x)u, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \Delta_{13}u(0) + (\Delta_{12} - \Delta_{14})u'(0) = 0, \\ u'\left(\frac{\pi}{2}\right) = 0; \end{cases} \tag{4.7}$$

If $y \in D(L)$, then $v(x) = Qy \in D(L^+)$, and

$$QLy = L^+Qy = L^+QQy = L_2v = L^+v = -v''(x) + \bar{q}(x)v = -v''(x) + q(x)v.$$

From $Qy \in D(L^+)$ it follows that

$$\overline{\Delta_{13}}\frac{y(0) - y(\pi)}{2} - (\overline{\Delta_{12}} + \overline{\Delta_{32}})\frac{y'(0) + y'(\pi)}{2} = 0,$$

$$\overline{\Delta_{13}}v(0) - (\overline{\Delta_{12}} + \overline{\Delta_{32}})v'(0) = 0,$$

$$v(0) - \frac{\overline{\Delta_{12} + \Delta_{32}}}{\overline{\Delta_{13}}}v'(0) = 0,$$

$$v(0) - \frac{\Delta_{12} + \Delta_{32}}{\Delta_{13}}v'(0) = 0,$$

$$\Delta_{13}v(0) - (\Delta_{12} + \Delta_{32})v'(0) = 0.$$

Thus,

$$L_2v = -v'' + q(x)v, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \Delta_{13}v(0) - (\Delta_{12} + \Delta_{32})v'(0) = 0, \\ v\left(\frac{\pi}{2}\right) = 0. \end{cases} \tag{4.8}$$

Equating coefficients of the boundary conditions (4.7) and (4.8), we have

$$\Delta_{12} - \Delta_{14} = -(\Delta_{12} + \Delta_{32}), => \ \Delta_{12} = \Delta_{14} - \Delta_{12} - \Delta_{32} =$$

$$= -(\Delta_{12} + \Delta_{32} - \Delta_{14}) = -\Delta_{34}.$$

Then the operators L_1 and L_2 have the following forms

$$L_1u = -u'' + q(x)u, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \Delta_{13}u(0) - (\Delta_{12} + \Delta_{32})u'(0) = 0, \\ u\left(\frac{\pi}{2}\right) = 0. \end{cases}$$

$$L_2v = -v'' + q(x)v, \qquad x \in \left(0, \frac{\pi}{2}\right),$$

$$\begin{cases} \Delta_{13}v(0) - (\Delta_{12} + \Delta_{32})v'(0) = 0, \\ v\left(\frac{\pi}{2}\right) = 0. \end{cases}$$

If spectrum of the operator L is known, then, by Lemma 4.2, proved earlier, spectra of the operators L_1 and L_2 are known. Then, by Borg theorem, the operator L_2 is uniquely defined on the interval $\left[0, \frac{\pi}{2}\right]$, and, due to parity and periodicity of the function $q(x)$, on the whole interval $[0, \pi]$.

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ПОТЕНЦИАЛЫ СИММЕТРИЯЛЫ, АЛ ШЕКАРАЛЫҚ ШАРТТАРЫ АЖЫРАМАЙТЫН ШТУРМ-ЛИУВИЛЛ ОПЕРАТОРЫНЫҢ КЕРІ ЕСЕБІ ТУРАЛЫ

Аннотация. Бұл еңбекте потенциалы симметриялы, нақты әрі үзіксіз, ал шекаралық шарттары ажырамайтын Штурм-Лиувилл операторын бір спектр арқылы анықтауға болатыны көрсетілді. Зерттеу әдісі бұрынғы әдістердің ешбіріне ұқсамайды, және ол оператордың ішкі симметриясына негізделген, ал ол өз кезегінде инвариантты кеңістіктердің салдары.

Түйін сөздер: Штурм-Лиувиллдің операторы, спектр, Штурм-Лиувиллдің кері есебі, Боргтың теоремасы, Амбарцумянның теоремасы, Левинсонның теоремасы, ажырамайтын шекаралық шарттар, симметриялы потенциал, инвариантты кеңістіктер.

УДК 517.9

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ОБРАТНАЯ ЗАДАЧА ОПЕРАТОРА ШТУРМА-ЛИУВИЛЛЯ С НЕ РАЗДЕЛЕННЫМИ КРАЕВЫМИ УСЛОВИЯМИ И СИММЕТРИЧНЫМ ПОТЕНЦИАЛОМ

Аннотация. В данной работе доказана теорема единственности, по одному спектру, для оператора Штурма-Лиувилля с не разделенными краевыми условиями и вещественным непрерывным и симметричным потенциалом. Метод исследования отличается от всех известных методов, и основан на внутреннюю симметрию оператора, порожденного инвариантными подпространствами.

Ключевые слова: Оператор Штурма-Лиувилля, спектр, обратная задача Штурма-Лиувилля, теорема Борга, теорема Амбарцумяна, теорема Левинсона, неразделенные краевые условия, симметричный потенциал, инвариантные подпространства.

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REFERENCES

[1] Marchenko V. A. Sturm – Liouville operators and their applications. Birkh¨auser, 1986.
[2] Levitan B. M. Inverse Sturm – Liouville problems.Utrecht, VNU Sci. Press, 1987.
[3] P¨oschel J., Trubowitz E. Inverse Spectral Theory.New York, Academic Press, 1987.
[4] Freiling G., Yurko V. A. Inverse Sturm – Liouville Problems and their Applications. New York, NOVA Science Publ., 2001.
[5] Yurko V. A. Method of Spectral Mappings inthe Inverse Problem Theory. Inverse and Ill-posed Problems Series. Utrecht, VSP, 2002.
[6] Stankevich I. V. An inverse problem of spectral analysis for Hill's equation. Soviet Math. Dokl., 1970, vol. 11, pp. 582–586.
[7] Marchenko V. A., Ostrovskii I. V. A characterization of the spectrum of the Hill operator. Math. USSR-Sb., 1975, vol. 26, no. 4, pp. 493–554. DOI:10.1070/SM1975v026n04ABEH002493.
[8] Yurko V. A. An inverse problem for second order differential operators with regular boundary conditions.Math. Notes, 1975, vol. 18, no. 3–4, pp. 928–932. DOI: 10.1007/BF01153046.

[9] Yurko V. A. On a periodic boundary value problem.Differ. Equations and Theory of Functions, Saratov, Saratov Univ. Press, 1981, pp. 109–115 (inRussian).

[10] Yurko V. A. On recovering differential operators with nonseparated boundary conditions. Study in Math. and Appl., Ufa, Bashkir Univ. Press, 1981,pp. 55–58 (in Russian).

[11] Plaksina O. A. Inverse problems of spectral analysis for the Sturm– Liouville operators with nonseparated boundary conditions. Math. USSRSb.,1988, vol. 59, no. 1, pp. 1–23. DOI:10.1070/SM1988v059n01ABEH003121.

[12] Guseinov I. M., Gasymov M. G., Nabiev I. M. An inverse problem for the Sturm– Liouville operator with nonseparable self-adjoint boundary conditions.Siberian Math. J., 1990, vol. 31, no. 6, pp. 910–918.

[13] Guseinov I. M., Nabiev I. M. Solution of a class of inverse boundary-value Sturm– Liouville problems.Sb. Math., 1995, vol. 186, no. 5, pp. 661–674. DOI:10.1070/SM1995v186n05ABEH000035.

[14] Kargaev P., Korotyaev E. The inverse problem for the Hill operator, a direct approach. Invent. Math., 1997, vol. 129, no. 3, pp. 567–593.

[15] Yurko V. A. On differential operators with nonseparated boundary conditions. Funct. Anal. Appl., 1994, vol. 28, no. 4, pp. 295–297. DOI: 10.1007/BF01076118.

[16] Yurko V. A. The inverse spectral problem for differential operators with nonseparated boundary conditions.J. Math. Analysis Appl., 2000, vol. 250,no. 1, pp. 266–289.

[17] Freiling G., Yurko V. A. On the stability of constructing a potential in the central symmetry case. Applicable Analysis, 2011, vol. 90, no. 12,pp. 1819–1828.

[18] V. A. Yurko.On the inverse periodic problem for centrally symmetric potentials. WPI. Sarat. UN-TA. New. ser. Ser. Mathematics. Mechanics. Informatics. 2016. Vol. 16, vol. 1,p. 68-75.

[19] Hald O.H. The Inverse Sturm--Liouville Problem with symmetric Potentials ,Acta mathematica 1978,141:1,263 .

[20] Isaakson E. L., Trubowitz - Pure Appl. Math. 1983. V. 36. N 6. P. 763-783.

[21] Isaakson E. L., McKean H. P., Trubowitz - Pure Appl. Math. 1984. V. 37. N 1. P. 1-12.

[22] Esin Inan Eskitascioglu, Mehmet Acil. An inverse Sturm-Liouville problem with a generalized symmetric potential., Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 41, pp. 17. ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu.

[23] Martin Bohner, Hikmet Koyunbakanb.Inverse Problems for Sturm–Liouville Di_erence Equations. Filomat 30:5 (2016), 1297–1304 DOI 10.2298/FIL1605297B.

[24] Münevver Tuz.On inverse Sturm- Liouville problems with symmetric potentials., ITM Web of Conferences **13**, 01035 (2017) DOI: 10.1051/itmconf/20171301035.

[25] Ambartsumyan V.A. Uber eine Frage der Eigenwert-theorie, Zsch.f.Physik, 53(1929), 690-695.

[26] Borg G. Eine Umbehrung der Sturm – Liovillschen Eigenwertaufgabe, Acta Math., 78, №2(1946), 1-96.

[27] Levinson N. The inverse Sturm – Lioville problem, Math.Tidsskr. B., 1949, 25-30.

[28] Levitan B.M. On definition of the Sturm - Liouville operator by two spectra. Proceedings of the USSR Academy of Sciences, ser. Math., 28 (1964), 63-78.

[29] Levitan B.M. On definition of the Sturm - Liouville operator by one and two spectra. Proceedings of the USSR Academy of Sciences, ser. Math., 42. No. 1. 1964.

[30] Akylbayev M.I., Beysebayeva A. and Shaldanbayev A. Sh., On the Periodic Solution of the Goursat Problem for a Wave Equation of a Special Form with Variable Coefficients. News of the National Academy of Sciences of the republic of Kazakhstan. Volume 1, Number 317 (2018), 34 – 50.

[31] Shaldanbaeva A. A., Akylbayev M.I., Shaldanbaev A. Sh. and Beisebaeva A.Zh., The Spectral Decomposition of Cauchy Problem's Solution for Laplace Equation. News of the National Academy of Sciences of the republic of Kazakhstan. Volume 5, Number 321 (2018), 75 – 87. https://doi.org/10.32014/2018.2518-1726.10

[32] T.Sh. Kal'menov and A.Sh. Shaldanbaev, On a criterion of solvability of the inverse problem of heat conduction. Journal of Inverse and Ill-posed Problems. 18:5, (2010). 471_492.

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RESEARCH OF MULTIPERIODIC SOLUTIONS OF PERTURBED LINEAR AUTONOMOUS SYSTEMS WITH DIFFERENTIATION OPERATOR ON THE VECTOR FIELD

Abstract. A linear system with a differentiation operator *D* with respect to the directions of vector fields of the form of the Lyapunov's system with respect to space independent variables and a multiperiodic toroidal form with respect to time variables is considered. All input data of the system multiperiodic depend on time variables or do not depend on them. The autonomous case of the system was considered in our early work. In this case, some input data received perturbations depending on time variables. We study the question of representing the required motion described by the system in the form of a superposition of individual periodic motions of rationally incommensurable frequencies. The initial problems and the problems of multiperiodicity of motions are studied. It is known that when determining solutions to problems, the system integrates along the characteristics outgoing from the initial points, and then, the initial data is replaced by the first integrals of the characteristic systems. Thus, the required solution consists of the following components: characteristics and first integrals of the characteristic systems of operator *D*, the matricant and the free term of the system itself. These components, in turn, have periodic and non-periodic structural components, which are essential in revealing the multiperiodic nature of the movements described by the system under study. The representation of a solution with the selected multiperiodic components is called the multiperiodic structure of the solution. It is realized on the basis of the well-known Bohr's theorem on the connection of a periodic function of many variables and a quasiperiodic function of one variable. Thus, more specifically, the multiperiodic structures of general and multiperiodic solutions of homogeneous and inhomogeneous systems with perturbed input data are investigated. In this spirit, the zeros of the operator D and the matricant of the system are studied. The conditions for the absence and existence of multiperiodic solutions of both homogeneous and inhomogeneous systems are established.

Keywords: multiperiodic solutions, autonomous system, operator of differentiation, Lyapunov's vector field, perturbation.

1. Introduction. The foundations of the method used in this note were laid in [1, 2], which were further developed in [3–14] and applied to the study of solutions different problems in the partial differential equations [15, 16]. These methods with simple modifications extend to the study solutions of problems of the differential and integro-differential equations of different types [1-16], in particular, problems on multi-frequency solutions of equations from control theory [17]. Many oscillatory phenomena are described by systems with a differentiation operator with respect to toroidal vector fields, and new methods based on the ideas of the Fourier [18], Poincaré-Lyapunov and Hamilton-Jacobi methods [19, 20] appear to establish their periodic oscillatory solutions. The methods of research for multiperiodic solutions are successfully combined by methods for studying solutions of boundary value problems for equations of mathematical physics. Elements of the methods of [1, 2] can easily be found in [21–25], where time-oscillating solutions of boundary value problems are studied by the parameterization method.

As noted above, the considered system of partial differential equations along with multidimensional time contains space independent variables, according to which differentiation is carried out to the directions of the different vector fields. The autonomous case of this system was considered in [15, 16], where differentiation with respect to time variables was carried out in the direction of the main diagonal of space, and the free term of the system was independent of time variables. In this case, these parameters of the systems received perturbations depending on time variables. In the note, the method for studying multiperiodic structures of general and multiperiodic solutions is developed, the conditions for the existence of a multiperiodic solution are established, and its integral representation is given.

We consider the system of linear equations

$$Dx = Ax + f(\tau,t,\zeta) \tag{1.1}$$

with differentiation operator

$$D = \frac{\partial}{\partial\tau} + \left\langle a, \frac{\partial}{\partial t}\right\rangle + \left\langle \nu I\zeta + g, \frac{\partial}{\partial\zeta}\right\rangle, \tag{1.2}$$

where $\tau \in (-\infty, +\infty) = R$, $t = (t_1, ..., t_m) \in R \times ... \times R = R^m$, $\zeta = (\zeta_1, ..., \zeta_l) \in R_\delta^{2l}$, $\zeta_j = (\xi_j, \eta_j)$, $j = \overline{1,l}$, $R_\delta^2 = \left\{\zeta_j = (\xi_j, \eta_j) \in R^2 : |\zeta_j| = \sqrt{\xi_j^2 + \eta_j^2} < \delta, j = \overline{1,l}\right\}$, $\delta = const > 0$ are independent variables with areas of change; $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_m}\right)$ and $\frac{\partial}{\partial\zeta} = \left(\frac{\partial}{\partial\zeta_1}, ..., \frac{\partial}{\partial\zeta_l}\right)$, $\frac{\partial}{\partial\zeta_j} = \left(\frac{\partial}{\partial\xi_j}, \frac{\partial}{\partial\eta_j}\right)$, $j = \overline{1,l}$ are vector differentiation operators; $I = diag\ (I_2, ..., I_2)$ is a matrix with l-blocks, I_2 is symplectic unit of the second order, $\nu = (\nu_1, ..., \nu_l)$ is a constant vector, $\nu I = diag\ (\nu_1 I_2, ..., \nu_l I_2)$, $a = (a_1(\tau,t), ..., a_m(\tau,t)) = a(\tau,t)$, $g = (g_1(\tau), ..., g_l(\tau)) = g(\tau)$ are vector functions, $\langle\ ,\ \rangle$ is the sign of the scalar product of vectors; A is a constant $n \times n$-matrix, $f = f(\tau,t,\zeta)$ is n-vector-function of variables $(\tau, t, \zeta) \in R \times R^m \times R_\delta^{2l}$.

The vector function $x(\tau,t,\zeta)$ is called (θ,ω)-periodic with respect to (τ,t) if the identity

$$x(\tau+\theta, t+q\omega, \zeta) = x(\tau,t,\zeta),\ (\tau,t,\zeta) \in R \times R^m \times R_\delta^{2l},\ q \in Z^m,$$

was fulfilled, where $Z^m = Z \times ... \times Z$, Z is the set of integers, $\omega = (\omega_1, ..., \omega_m)$ is the vector-period, and the periods $\omega_0 = \theta, \omega_1, ..., \omega_m$ are rationally incommensurable positive constants:

$$q_j\omega_j + q_k\omega_k \neq 0,\ \ q_j, q_k \in Z,\ (j, k = \overline{0,m}).$$

The motion described by a (θ,ω)-periodic with respect to (τ,t) function $x = x(\tau,t,\zeta)$ is called a multiperiodic oscillation.

The main objective of this note is to determine the multiperiodic structures of solutions of the initial-multiperiodic problems associated with the system (1.1) - (1.2).

The objective was partially been touched upon by the authors in [15, 16], when the problem of multiperiod solutions of the autonomous system of the form (1.1) - (1.2) was considered, where time variables τ, t did not explicitly enter.

2. Multiperiodic structure of zeros of the differentiation operator D. We introduce the equation

$$Du = 0 \tag{2.1}$$

with the required scalar function $u = u(\tau,t,\zeta)$, where D is the differentiation operator with respect to (τ,t,ζ) of the form (1.1).

The solutions of equation (2.1) are called the zeros of the operator D .

Suppose that 1) the vector function $a(\tau,t)$ has the property of smoothness with respect to $(\tau,t)\in R\times R^m$ of order $(0,e)=(0,1,...,1)$:

$$a(\tau+\theta,t+q\omega)=a(\tau,t)\in C_{\tau,t}^{(0,e)}(R\times R^m),\ q\in Z^m, \tag{2.2}$$

2) positive constants ν_1, ..., ν_l rationally incommensurable:

$$q_i\nu_i+q_j\nu_j\neq 0,\ q_i^2+q_j^2\neq 0,\quad q_i,q_j\in Z,\ (i,\ j=\overline{0,l}), \tag{2.3}$$

therefore, numbers $\alpha_j=2\pi\nu_j^{-1},\ j=\overline{1,l}$ are also incommensurable.

3) vector-functions $g_j(\tau)=(\varphi_j(\tau),\psi_j(\tau)),\ j=\overline{1,l}$ are continuous and β_j-periodic:

$$g_j(\tau+\beta_j)=g_j(\tau)\in C_\tau^{(0)}(R),\ j=\overline{1,l}, \tag{2.4}$$

where $\alpha_k,\ k=\overline{1,l}$ and $\beta_j,\ j=\overline{1,l}$ are incommensurable positive constants.

It follows from condition (2.2) that the vector field

$$\frac{dt}{d\tau}=a(\tau,t) \tag{2.5}$$

determines the characteristic

$$t=\lambda(\tau,\tau^0,t^0), \tag{2.5^1}$$

emanating from any initial point $(\tau^0,t^0)\in R\times R^m$, and moreover, it has the properties:

$$t^0=\lambda(\tau^0,\tau,t), \tag{2.5^2}$$

$$\lambda(\tau',\tau'',\lambda(\tau'',\tau,t))=\lambda(\tau',\tau,t),\ \tau',\tau''\in R, \tag{2.5^3}$$

$$\lambda(\tau^0+\theta,\tau+\theta,t+q\omega)=\lambda(\tau^0,\tau,t)+q\omega,\ q\in Z^m, \tag{2.5^4}$$

$$DV(\lambda(\tau^0,\tau,t))=0,\ V(t)\in C_t^{(e)}(R^m). \tag{2.5^5}$$

Obviously, $u=v(\lambda(\tau^0,\tau,t))$ satisfies the initial condition

$$u\big|_{\tau=\tau^0}=v(t)\in C_t^{(e)}(R^m). \tag{2.1'}$$

Properties (2.5²) - (2.5⁵) of the characteristic (2.5¹) of the vector field (2.5) are known from [2]. Hence, we will not dwell on their justification.

The solution

$$u(\tau^0,\tau,t)=v(\lambda(\tau^0,\tau,t)) \tag{2.6}$$

of the problem (2.1) - (2.1 ') is called the zero of the operator D with the initial condition (2.1').

Lemma 2.1. *Let condition (2.2) be satisfied. Then under the condition*

$$v(t+q\omega)=v(t)\in C_t^{(e)}(R^m),\ q\in Z^m \tag{2.7}$$

the zeros (2.6) of the operator D with the initial data (2.1') have the multiperiodicity property of the form

$$u(\tau^0+\theta,\tau+\theta,t+q\omega)=u(\tau^0,\tau,t),\ q\in Z^m. \tag{2.8}$$

The proof of identity (2.8) follows from the structure of zero (2.6), property (2.5⁴) which is a consequence of condition (2.2), and from condition (2.7).

Note that property (2.8) represent the diagonal θ-periodicity $u(\tau^0,\tau,t)$ with respect to (τ^0,τ) and ω-periodicity with respect to t.

In particular, when a function $\lambda(\tau^0,\tau,t)$ is θ-periodic with respect to τ or τ^0, then the zeros (2.6) of the operator D under the conditions of the lemma are (θ,ω)-periodic with respect to (τ,t).

The vector fields

$$\frac{d\zeta_j}{d\tau}=\nu_j I_2\zeta_j+g_j(\tau),\ j=\overline{1,l} \tag{2.9}$$

in scalar form have the form

$$\begin{cases}\dfrac{d\xi_j}{d\tau}=-\nu_j\eta_j+\varphi_j(\tau),\\ \dfrac{d\eta_j}{d\tau}=\nu_j\xi_j+\psi_j(\tau),\ \ j=\overline{1,l}.\end{cases} \tag{2.10}$$

Obviously, the matricants $Z_j(\tau),\ j=\overline{1,l}$ of the systems (2.10), and, consequently, the systems (2.9), are determined by periodic relations

$$Z_j(\tau)=\begin{pmatrix}\cos\nu_j\tau & -\sin\nu_j\tau\\ \sin\nu_j\tau & \cos\nu_j\tau\end{pmatrix},\ j=\overline{1,l} \tag{2.11}$$

with periods $\alpha_j=2\pi\nu_j^{-1},\ j=\overline{1,l}$. The conditions

$$\det\left[Z_j(\beta_j)-Z_j(0)\right]\neq 0,\ j=\overline{1,l}. \tag{2.12}$$

are satisfied by virtue the incommensurability α_k and β_j. Indeed

$$\det\left[Z_j(\beta_j)-Z_j(0)\right]=2\left(1-\cos\nu_j\beta_j\right)\neq 0\ ,$$

since $\beta_j-q_j\alpha_j\neq 0,\ \ j=\overline{1,l}$.

Then systems (2.9) allow for β_j-periodic solutions

$$z_j(\tau)=\left[Z_j^{-1}(\tau+\beta_j)-Z_j^{-1}(\tau)\right]^{-1}\int\limits_{\tau}^{\tau+\beta_j}Z_j^{-1}(s)\,g_j(s)ds,\ j=\overline{1,l}. \tag{2.13}$$

Consequently, the general solutions ζ_j of the systems (2.9) have the form

$$\zeta_j=Z_j\left(\tau-\tau^0\right)\left[\zeta_j^0-z_j\left(\tau^0\right)\right]+z_j\left(\tau\right),\ j=\overline{1,l}, \tag{2.14}$$

where the matricants $Z_j\left(\tau\right),\ j=\overline{1,l}$ and solutions $z_j(\tau),\ \ j=\overline{1,l}$ have periodicity properties

$$Z_j\left(\tau+\alpha_j\right)=Z_j\left(\tau\right),\ \ j=\overline{1,l}, \tag{2.15}$$

$$z_j\left(\tau+\beta_j\right)=z_j\left(\tau\right),\ \ j=\overline{1,l}. \tag{2.16}$$

We must introduce new time variables $s_j,\ \sigma_j, j=\overline{1,l}$ and space variables $h_j, j=\overline{1,l}$ related by relations

$$h_j\left(s_j-s_j^0,\sigma_j,\ \zeta_j^0-z_j^0\right)=Z_j\left(s_j-s_j^0\right)\left[\zeta_j^0-z_j^0\right]+z_j\left(\sigma_j\right),\ j=\overline{1,l}, \tag{2.17}$$

in order to represent solutions (2.14) using periodic functions with incommensurable periods $\alpha_j,\ \beta_j, j=\overline{1,l}$, where $z_j^0=z_j\left(s_j^0\right),\ s_j^0$ are the initial values of the variables $s_j,\ j=\overline{1,l}$.

Obviously, the multiperiodic functions (2.17) present the solutions (2.14) under $\sigma_j=s_j=\tau,\ \ s_j^0=\tau^0$, moreover, they satisfy equations

$$\frac{\partial h_j}{\partial s_j}+\frac{\partial h_j}{\partial \sigma_j}=\nu_j I_2 h_j+g_j(\sigma_j)\quad j=\overline{1,l} \tag{2.18}$$

with the initial conditions

$$h_j\big|_{\sigma_j=s_j=s_j^0}=\zeta_j^0,\ \ j=\overline{1,l}. \tag{2.18°}$$

By virtue of the properties (2.15) and (2.16) of the matricants $Z_j(\tau)$ and the solutions $z_j(\tau)$, the functions (2.17) have the properties of multi-periodicity

$$h_j\left(s_j+\alpha_j,\sigma_j,\zeta_j^0\right)=h_j\left(s_j,\sigma_j+\beta_j,\zeta_j^0\right)=h_j\left(s_j,\sigma_j,\zeta_j^0\right),\ j=\overline{1,l}. \tag{2.19}$$

Thus, we obtained from systems of equations (2.9) to systems of equations (2.18) with initial conditions (2.18°) by introducing new time variables.

We get the equations (2.9) and their solutions (2.14) from the systems of equations (2.18) - (2.18°) by substitution $\sigma_j=s_j=\tau,\ \ s_j^0=\tau^0$ conversely.

The close relationship between the functions $\sigma_j=\sigma_j(\tau)$ and $h_j=h_j(s_j,\sigma_j)$ of the form

$$\sigma_j(\tau)=h_j(\tau,\tau),\quad \frac{d\sigma_j}{d\tau}=\frac{dh_j(\tau,\tau)}{d\tau}=\frac{\partial h_j(s_j,\sigma_j)}{\partial s_j}+\frac{\partial h_j(s_j,\sigma_j)}{\partial \sigma_j}$$

with $\sigma_j=s_j=\tau$ leads to a transition from the differentiation operator D to the differentiation operator

$$\overline{D}=\frac{\partial}{\partial\tau}+\left\langle a(\tau,t),\frac{\partial}{\partial t}\right\rangle+\left\langle e,\frac{\partial}{\partial s}\right\rangle+\left\langle e,\frac{\partial}{\partial\sigma}\right\rangle+\left\langle \nu I h+g(\sigma),\frac{\partial}{\partial h}\right\rangle+\left\langle \frac{\partial h}{\partial s}+\frac{\partial h}{\partial \sigma},\frac{\partial}{\partial h}\right\rangle, \tag{2.20}$$

where $s=(s_1,...,s_l),\ \sigma=(\sigma_1,...,\sigma_l),\ e=(1,...,1)$ – l-vector, $h=(h_1,...,h_l),\ \ h_j=h_j(s_j,\sigma_j),$ $j=\overline{1,l},\ \frac{\partial h}{\partial s}=\left(\frac{\partial h_1}{\partial s_1},...,\frac{\partial h_l}{\partial s_l}\right),\ \frac{\partial h}{\partial \sigma}=\left(\frac{\partial h_1}{\partial \sigma_1},...,\frac{\partial h_l}{\partial \sigma_l}\right).$

Further, we obtain the characteristic

$$\zeta=Z(\tau-\tau^0)\left[\zeta^0-z(\tau^0)\right]+z(\tau) \tag{2.21}$$

of the matrix-vector equation

$$\frac{d\zeta}{d\tau}=\nu I\zeta+g(\tau), \tag{2.22}$$

which is characteristic for equation (2.1) with respect to space variables, based on the coordinate data (2.9) - (2.16), where $Z(\tau)=diag\ \left[Z_1(\tau),...,Z_l(\tau)\right],\ z(\tau)=\left(z_1(\tau),...,z_l(\tau)\right),\ \zeta^0=\left(\zeta_1^0,...,\zeta_l^0\right).$

We have the first integral

$$\zeta^0=Z(\tau^0-\tau)\left[\zeta-z(\tau)\right]+z(\tau^0)\equiv\mu(\tau^0,\tau,\zeta) \tag{2.23}$$

of equation (2.22) from the equation of characteristic (2.21).

Therefore, we obtain the identity

$$D\mu(\tau^0,\tau,\zeta)=0,\quad \mu(\tau^0,\tau^0,\zeta)=\zeta. \tag{2.24}$$

Then we have the solution

$$u(\tau^0,\tau,\zeta)=w(\mu(\tau^0,\tau,\zeta)), \tag{2.25}$$

of equation (2.1) satisfying the initial condition

$$u\big|_{\tau=\tau^0}=w(\zeta)\in C_\zeta^{(e)}(R^l). \tag{2.1''}$$

for any differentiable function $w(\zeta)\in C_\zeta^{(e)}(R^l)$.

Indeed, since $Du = \frac{\partial w}{\partial \zeta} \cdot D\mu$, by virtue of (2.24) we have $Du = 0$. Thus, (2.25) with the condition (2.1'') is the zero of the operator D.

Further, we have a vector function

$$h\left(s - s^0, z(\sigma), \zeta^0 - z^0\right) = Z\left(s - s^0\right)\left[\zeta^0 - z\left(s^0\right)\right] + z(\sigma), \tag{2.26}$$

satisfying the characteristic equation of the operator $\overline{D}$ of the form

$$\frac{\partial h}{\partial s} + \frac{\partial h}{\partial \sigma} = \nu I h + g(\sigma) \tag{2.27}$$

with the initial condition

$$h\big|_{\sigma = s = s^0} = \zeta^0,$$

based on our analysis related to relations (2.17) - (2.19) for studying the multi-periodic structure of characteristic (2.23), where $g(\sigma) = \left(g_1(\sigma_1), ..., g_l(\sigma_l)\right)$, $z(\sigma) = \left(z_1(\sigma_1), ..., z_l(\sigma_l)\right)$, $Z(s) = diag\ \left[Z_1(s_1), ..., Z_l(s_l)\right]$, $h = (h_1, ..., h_l)$, $h_j = h_j\left(s_j - s_j^0, z(\sigma_j^0), \zeta_j^0 - z(s^0)\right)$, $j = \overline{1, l}$, $\frac{\partial h}{\partial s} = \left(\frac{\partial h_1}{\partial s_1}, ..., \frac{\partial h_l}{\partial s_l}\right)$, $\frac{\partial h}{\partial \sigma} = \left(\frac{\partial h_1}{\partial \sigma_1}, ..., \frac{\partial h_l}{\partial \sigma_l}\right)$.

Obviously, by virtue properties (2.15), (2.16) and (2.19), the matrix $Z(s)$ is periodic with period $\alpha = (\alpha_1, ..., \alpha_l)$, and the solution $z(\sigma)$ with period $\beta = (\beta_1, ..., \beta_l)$.

The first integral of the equation (2.27) is determined from the equation of characteristic (2.26) by the relation

$$\zeta^0 = h\left(s^0 - s, z(s^0), \zeta - z(\sigma)\right).$$

It's obvious that

$$\overline{D} h\left(s^0 - s, z(s^0), \zeta - z(\sigma)\right) = 0,\ \ h\big|_{\sigma = s = s^0} = \zeta. \tag{2.28}$$

Moreover, we have

$$\overline{D} w\left(h\left(s^0 - s, z(s^0), \zeta - z(\sigma)\right)\right) = \frac{\partial w(h)}{\partial \zeta} \cdot \overline{D} h\left(s^0 - s, z(s^0), \zeta - z(\sigma)\right) = 0,$$

for any differentiable function $w(\zeta)$, by virtue of (2.28), at that

$$w\left(h\left(s^0 - s, z(s^0), \zeta - z(\sigma)\right)\right)\big|_{\sigma = s = s^0} = w(\zeta).$$

Thus,

$$\overline{u}\left(s^0, s, \sigma, \zeta\right) = w\left(h\left(s^0 - s, z(s^0), \zeta - z(\sigma)\right)\right) \tag{2.29}$$

is the zero of the operator $\overline{D}$, that under $\sigma = s = \tau \widetilde{e},\ s^0 = \tau^0 \widetilde{e}$ it becomes the $u\left(\tau^0, \tau, \zeta\right)$ zero of the operator D, where $\widetilde{e} = (1, ..., 1)$ is a l-vector.

Lemma 2.2. *Let conditions (2.3) and (2.4) be satisfied. Then the zeros (2.25) of the operator D with the initial condition (2.1'') have a multiperiodic structure of the form (2.29) with the vector function (2.26), at that*

$$\overline{u}\left(s^0, s, \sigma, \zeta\right)\Big|_{\substack{\sigma = s = \tau \widetilde{e} \\ s^0 = \tau^0 \widetilde{e}}} = u\left(\tau^0, \tau, \zeta\right),$$
$$h\left(\widetilde{e}\tau^0 - \widetilde{e}\tau, z(\widetilde{e}\tau^0), \zeta - z(\widetilde{e}\tau)\right) = \mu\left(\tau^0, \tau, \zeta\right). \tag{2.30}$$

Theorem 2.1. *Let conditions (2.2) - (2.4) be satisfied. Then the solution* $u\left(\tau^0,\tau,t,\zeta\right)$ *of equation (2.1) with the initial condition*

$$u\Big|_{\tau=\tau^0} = u^0(t,\zeta) \in C_{t,\zeta}^{(\hat{e},\tilde{e})}\left(R^m \times R^l\right) \tag{2.1°}$$

is determined by the relation

$$u\left(\tau^0,\tau,t,\zeta\right) = u^0\left(\lambda\left(\tau^0,\tau,t\right), \mu\left(\tau^0,\tau,\zeta\right)\right), \tag{2.31}$$

which under the conditions

$$\lambda\left(\tau^0,\tau+\theta,t\right) = \lambda\left(\tau^0,\tau,t\right), \tag{2.32}$$

$$u^0(t+q\omega,\zeta) = u^0(t,\zeta),\ q \in Z^m \tag{2.33}$$

has a multiperiodic structure with respect to (τ,t,s,σ) *with period* $(\theta,\omega,\alpha,\beta)$ *of the form*

$$\overline{u}\left(\tau^0,\tau,t;s^0,s,\sigma,\zeta\right) = u^0\left(\lambda\left(\tau^0,\tau,t\right), h\left(s^0-s, z\left(s^0\right), \zeta - z(\sigma)\right)\right), \tag{2.34}$$

where the vector-function $h(s,z,\zeta)$ *has the form (2.26),* $\hat{e}=(1,...,1)$ *is* m*-vector,* $\tilde{e}=(1,...,1)$ *is* l*-vector, moreover*

$$\overline{u}\Big|_{\substack{\sigma=s=\tilde{e}\,\tau \\ s^0=\tilde{e}\,\tau^0}} = u\left(\tau^0,\tau,t,\zeta\right). \tag{2.35}$$

Proof. The form of solution (2.31) of the initial problem (2.1) - (2.1°) follows from the general theory of the first-order partial differential equations. Special cases of it are given in Lemmas 2.1 and 2.2.

The multiperiodic structure (2.34) of the solution (2.31) is also contained in the indicated lemmas; and the multiperiodicity is easily verified under the additional conditions (2.32) and (2.33).

The statement (2.35) follows from (2.30).

Note that, $\overline{u} = \overline{u}\left(\tau^0,\tau,t,s^0,s,\sigma,\zeta\right)$ is the solution of the equation $\overline{D}\overline{u}=0$ with the differentiation operator $\overline{D}$.

The proved theorem is the multiperiodic structure of the zeros of the differentiation operator D.

In conclusion, we note that if the conditions (2.32) and (2.33) do not fulfill, then the representation (2.34) remains the multi-periodic structure of the solution (2.31). But then a definite structure (2.34) does not possess the periodicity property with respect to τ, t.

3. The multiperiodic structure of the solution of a homogeneous linear D **-system with constant coefficients.** We consider a homogeneous linear system

$$Dx = Ax \tag{3.1}$$

with a differentiation operator D of the form (1.2) and a constant $n \times n$-matrix A.

We will put the problem of determining the multiperiodic structure of the solution x of the system (3.1) with the initial condition

$$x\Big|_{\tau=\tau^0} = u(t,\zeta) \in C_{t,\zeta}^{(\hat{e},\tilde{e})}\left(R^m \times R^l\right). \tag{3.1°}$$

To this end, we begin the solution of the problem by studying the multiperiodic structure of the matricant

$$X(\tau) = \exp\left[A\,\tau\right] \tag{3.2}$$

of the system (3.1).

We need the following lemmas to that end.

Lemma 3.1. *If* $f_j\left(\tau+\theta_j\right) = f_j(\tau),\ j=\overline{1,r}$ *is some collection of the periodic functions with rationally commensurate periods:* $\theta_j/\theta_k = r_{jk}$ *is a rational number for* $j,k=\overline{1,r}$*, then for these functions exist a common period* θ*:*

$$f_j(\tau+\theta) = f_j(\tau),\ j=\overline{1,r}.$$

Indeed, by virtue of rational commensurability exist integer natural numbers $q_1,..., q_r$ such that $q_1\theta_1 = ... = q_r\theta_r = \theta$, which is the required period.

Lemma 3.2. *If the real parts of all eigenvalues equal to zero and all the elementary divisors are simple of the constant matricant* $Y(\tau) = \exp[I\tau]$, *then all the elements of the matrix* I *are periodic functions*.

Proof. By the conditions of the Lemma 3.2, the eigenvalues are $\lambda_j(I) = ib_j$, $j = \overline{1, r}$, where $i = \sqrt{-1}$ is the imaginary unit; the constants b_j are either equal to zero or nonzero. If it is nonzero, then each eigenvalue $\lambda_j(I) = ib_j$ corresponds to one or more Jordan cells J_j of the form

$$J_j = \begin{pmatrix} 0 & -b_j \\ b_j & 0 \end{pmatrix}.$$

Then the matricant has the form

$$Y(\tau) = K\, diag\left[e^{I_1\tau},..., e^{I_r\tau}\right]K^{-1}, \tag{3.3}$$

where if $b_j = 0$, then $I_j = 0$ and if $b_j \neq 0$, then $I_j = J_j$, moreover

$$Y_j(\tau) = e^{J_j\tau} = \begin{pmatrix} \cos b_j\tau & -\sin b_j\tau \\ \sin b_j\tau & \cos b_j\tau \end{pmatrix}, \; (b_j \neq 0), \tag{3.4}$$

K is a matrix of reduction I to the actual canonical form $I = K\, diag\,[I_1,..., I_r]K^{-1}$.

We have a complete proof of the Lemma 3.2 from relations (3.3) and (3.4), and the periods of the elements of the matrix $Y(\tau)$ are determined as $\gamma_1 = 2\pi b_{j_1}^{-1},..., \gamma_\rho = 2\pi b_{j_\rho}^{-1}$ on the basis of the Lemma 3.1, taking into account the commensurability of the periods $2\pi b_j^{-1}$, $j = \overline{1, r}$, $\rho \leq r$ Periods $\gamma_1,...,\gamma_\rho$ are rationally incommensurable constants.

Further, cells $Y_{j_k}(\tau)$, $j_k = \overline{1, r_k}$ of the form (3.4) having the periodicity property with a period γ_k will be considered as cells depending on the variable $\tau = \tau_k$:

$$Y_{j_k}(\tau_k + \gamma_k) = Y_{j_k}(\tau_k), \; j_k = \overline{1, r_k}. \tag{3.5}$$

Representing each cell (3.4) using the new variables $\tau_1,..., \tau_\rho$ in accordance with condition (3.5), from the expression of the matricant (3.3) we obtain a multiperiodic matrix $T(\hat{\tau}) = T(\tau_1,..., \tau_\rho)$ with period $\gamma = (\gamma_1,..., \gamma_\rho)$.

Since

$$\frac{\partial}{\partial \tau_k} Y_{j_k}(\tau_k) = J_j Y_{j_k}(\tau_k),$$

the matrix $T(\hat{\tau})$ satisfies the equation

$$\hat{D}T(\hat{\tau}) = I\, T(\hat{\tau}), \tag{3.6}$$

where the operator $\hat{D}$ is determined by

$$\hat{D} = \left\langle \hat{e}, \frac{\partial}{\partial \hat{\tau}} \right\rangle = \frac{\partial}{\partial \tau_1} + ... + \frac{\partial}{\partial \tau_\rho}, \tag{3.7}$$

$\hat{e} = (1,..., 1)$ is a ρ-vector.

Obviously, under $\hat{\tau} = \hat{e}\tau$ we have $T(\hat{e}\tau) = Y(\tau)$ and

$$\frac{d}{d\tau}Y(\tau)=\frac{d}{d\tau}T(\hat{e}\,\tau)=I\,T(\hat{e}\,\tau)=I\,Y(\tau). \tag{3.8}$$

Thus, the multiperiodic matrix $T(\hat{\tau})$ defines the multiperiodic structure of the matricant $Y(\tau)$:

$$Y(\tau)=T(\tau_1,\dots,\tau_\rho)\Big|_{\tau_1=\dots=\tau_\rho=\tau}. \tag{3.9}$$

Lemma 3.3. *The matricant $Y(\tau)$ of the system (3.8) under the conditions of Lemma 3.2 has a multiperiodic structure in the form of a matrix $T(\hat{\tau})=T(\tau_1,\dots,\tau_\rho)$, which satisfies the system (3.6) with the differentiation operator (3.7) and along the characteristics $\hat{\tau}=\hat{e}\,\tau$ of the operator $\hat{D}$ turns into $Y(\tau)$, in other words, these matrices are related by the relation (3.9).*

It's known that from the course of linear algebra the matrix A can be represented in the form

$$A=K\,J(\lambda)K^{-1}=K\,J(a+ib)K^{-1}=K\,J(a)K^{-1}+K\,E(ib)K^{-1}=R+I,$$

where K is some non-singular matrix for reducing the matrix A to Jordan normal form $J(\lambda)=diag\,[J_1(\lambda_1),\dots,J_r(\lambda_r)]$ with Jordan's n_j-cells $J_j(\lambda_j)$ corresponding to eigenvalues $\lambda_j=a_j+ib_j$, $j=\overline{1,r}$; $R=K\,J(a)K^{-1}$ is the matrix, $J(a)$ is matrix obtained from the Jordan form $J(\lambda)$ by replacing the eigenvalues λ_j with their real parts $a_j=\mathrm{Re}\,\lambda_j$, $j=\overline{1,r}$, $I=K\,E(ib)K^{-1}$ is the matrix, $E(ib)=diag[ib_1E_1,\dots,ib_rE_r]$, $b_j=\mathrm{Im}\lambda_j$, $j=\overline{1,r}$, E_j is the unit n_j-cells, $j=\overline{1,r}$, moreover, the matrices R and I are commutative: $RI=IR.$. Therefore, $e^{A\tau}=e^{I\tau+R\tau}=e^{I\tau}\cdot e^{R\tau}$, otherwise, the matricant (3.2) can be represented as

$$X(\tau)=Y(\tau)\cdot Z(\tau), \tag{3.10}$$

where $Y(\tau)=\exp[I\tau], Z(\tau)=\exp[R\tau]$, moreover, along with property (3.8), $Y(\tau)$ satisfies the equation

$$\frac{d}{d\tau}Y(\tau)=AY(\tau)-Y(\tau)R. \tag{3.11}$$

Indeed, we making the replacement

$$X=Y(\tau)Z$$

in the equation

$$\dot{X}=AX \tag{3.12}$$

obtain the equation

$$\dot{Z}=Y^{-1}(\tau)\left[AY(\tau)-\frac{d}{d\tau}Y(\tau)\right]Z.$$

Then, we obtain the identity (3.10) taking into account that $\dot{Z}=RZ$, where $Z(\tau)=\exp[R\tau]$.

The identities (3.8) and (3.11) establish the connection of the matricant $Y(\tau)=\exp[I\tau]$ with the triple of matrices A,R,I; moreover, the matrix I satisfies the conditions of Lemma 3.2. Therefore, according to Lemma 3.3, the multiperiodic structure of the matricant $X(\tau)=\exp[A\tau]$, by virtue of equality (3.10), is determined by a matrix $\hat{X}(\tau,\hat{\tau})$ of the form

$$\hat{X}(\tau,\hat{\tau})=X(\tau,\tau_1,\dots,\tau_\rho)=T(\tau_1,\dots,\tau_\rho)e^{R\tau}, \tag{3.13}$$

which is connected by the matricant $X(\tau)$, by relation

$$\hat{X}(\tau,\hat{\tau})\Big|_{\hat{\tau}=\hat{e}\tau} = X(\tau). \tag{3.14}$$

Thus, the following theorem is proved.

Theorem 3.1. *In the presence of complex eigenvalues of the matrix A, the matricant (3.2) of the system (3.12) has a multiperiodic structure defined by the matrix (3.13) and relations (3.6) - (3.9), and it along the characteristics $\hat{\tau} = \hat{e}\tau$ of the operator $\hat{D}$ satisfies condition (3.14). The matrix $T(\hat{\tau})$ turns into a constant matrix in the absence of complex eigenvalues.*

Now the solution of the objectives set can be formulated as Theorem 3.2.

Theorem 3.2. *Let conditions (2.2) - (2.4) be satisfied. Then the solution $x(\tau^0,\tau,t,\zeta)$ of the problem (3.1) - (3.1°) defined by relation*

$$x(\tau^0,\tau,t,\zeta) = X(\tau)u(\lambda(\tau^0,\tau,t),\mu((\tau^0,\tau,\zeta))) \tag{3.15}$$

has a multi-periodic structure in the form of a vector-function

$$\hat{x}(\tau^0,\tau,\hat{\tau},t,\ s^0 s,\sigma,\zeta) = \hat{X}(\tau,\hat{\tau})\overline{u}(\lambda(\tau^0,\tau,t),h(s^0-s,z(s^0),\zeta-z(\sigma))), \tag{3.16}$$

that satisfies equation

$$\overline{\overline{D}}\hat{x} = A\hat{x} \tag{3.17}$$

with the differentiation operator

$$\overline{\overline{D}} = \overline{D} + \hat{D}, \tag{3.18}$$

defined by relations (2.20) and (3.7).

Proof. The representation (3.15) is known from [2], and (3.16) follows from the proved Theorems 2.1 and 3.1. The identity (3.17) can be verified by a simple check.

Now we investigate the question of the existence of nonzero multiperiodic solutions of the systems of equations (3.1).We begin the study with the simplest cases.

We consider a canonical system with a single zero eigenvalue

$$\frac{dx_1}{d\tau} = 0,\ \frac{dx_2}{d\tau} = x_1,\ldots,\frac{dx_n}{d\tau} = x_{n-1},$$

which in the vector-matrix form has the form

$$\frac{dx}{d\tau} = E_* x, \tag{3.19}$$

where E_* is the sub-diagonal unit oblique series of the n-th order, $x = (x_1,\ldots,x_n)$.

We introduce a triangular matrix $X_0(\tau)$ with elements of the form of power functions:

$$X_0(\tau) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ \tau & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{\tau^{n-1}}{(n-1)!} & \dfrac{\tau^{n-2}}{(n-2)!} & \ldots & 1 \end{pmatrix}$$

and an arbitrary constant vector $c = (c_1,\ldots,c_n)$ to represent the general solution x of the system (3.19).

Then the general solution of the system (3.19) is represented in the form $x = X_0(\tau)c$.

It easy to see from the structure of the general solution that system (3.19) admits a one-parameter family of periodic solutions x^* of the form

$$x^*(\tau) = X_0(\tau)c^*, \tag{3.20}$$

where $c^* = (0,\ldots,0,c_n^*)$, c_n^* is an arbitrary parameter.

Next, we consider a system of pairs $\left(x'_j, x''_j\right)$ of equations of the form

$$\frac{dx'_1}{d\tau} = -bx''_1, \ \frac{dx''_1}{d\tau} = bx'_1, \ \frac{dx'_j}{d\tau} = x'_{j-1} - bx''_j, \ \frac{dx''_j}{d\tau} = x''_{j-1} + bx'_j, \ j = \overline{1,l},$$

which can be represented using the vector $x_j = \left(x'_j, x''_j\right)$ in the form

$$\frac{dx_1}{d\tau} = bI_2 x_1, \ \frac{dx_j}{d\tau} = E_2 x_j + bI_2 x_j, \ j = \overline{1,l},$$

where E_2 is the second-order identity matrix, I_2 is the second-order symplectic identity matrix, $b = const \neq 0$.

If we introduce a constant block matrix

$$J(b) = \begin{pmatrix} bI_2 & O & O & ... & O & O & O \\ E_2 & bI_2 & O & ... & O & O & O \\ O & E_2 & bI_2 & ... & O & O & O \\ ... & ... & ... & ... & ... & ... & ... \\ O & O & O & ... & E_2 & bI_2 & O \\ O & O & O & ... & O & E_2 & bI_2 \end{pmatrix}$$

with blocks I_2, E_2 and second-order zero blocks O, then the system under consideration with a vector $x = (x_1, ..., x_l)$ can be represented in the form

$$\frac{dx}{d\tau} = J(b)x, \tag{3.21}$$

which we call a canonical system with a single pair of purely imaginary conjugate eigenvalues $\lambda = (ib, -ib)$.

We introduce a diagonal block matrix

$$T^*(\tau) = diag\,\left[T_2(\tau), ..., T_2(\tau)\right]$$

with a block $T_2^*(\tau)$ of the form

$$T_2^*(\tau) = \begin{pmatrix} \cos b\tau & -\sin b\tau \\ \sin b\tau & \cos b\tau \end{pmatrix}$$

and a triangular block matrix with elements of the form of power functions:

$$Y^*(\tau) = \begin{pmatrix} E_2 & O & ... & O \\ \tau E_2 & E_2 & ... & O \\ ... & ... & ... & ... \\ \dfrac{\tau^{l-1}}{(l-1)!}E_2 & \dfrac{\tau^{l-2}}{(l-2)!}E_2 & ... & E_2 \end{pmatrix}$$

to represent the general solution x of the system (3.21).

Then the matricant $X^*(\tau)$ of the system (3.21) can be represented as $X^*(\tau) = T^*(\tau)Y^*(\tau)$, and the general solution $x(\tau)$ is determined by the relation

$$x(\tau) = X^*(\tau)c$$

with an arbitrary constant vector $c = (c_1, ..., c_l),\ c_j = (c'_j,\ c''_j),\ j = \overline{1,l}$.

We obtain easily a family of $\theta = 2\pi b^{-1}$-periodic solutions $x^*(\tau)$ by parameters c'_l and c''_l of the form

$$x^*(\tau) = X^*(\tau)c^* \tag{3.22}$$

with a constant vector $c^* = (0, ..., 0, c_l), c_l = (c'_l,\ c''_l)$ from the structure of the general solution

Now, by replacing $x = Kz$ with a non-singular constant matrix K, we reduce the system (3.1) to the canonical form

$$Dz = J(A)z,\ J(A) = K^{-1}AK, \tag{3.1'}$$

which consists of subsystems in accordance with Jordan's cells of the matrix A.

Obviously, systems (3.1) and (3.1') are equivalent with respect to the existence of multiperiodic solutions.

It is also clear that the system (3.1') has subsystems of the form

$$Dz_1 = E_* z_1, \tag{3.1'_1}$$

or

$$Dz_2 = J(b)z_2, \tag{3.1'_2}$$

respectively with matrices similar to the matrices of systems (3.19) and (3.21), in the presence of zero or purely imaginary eigenvalue. Obviously, nonzero solutions of (3.20) and (3.22) satisfy the systems $(3.1'_1)$ and $(3.1'_2)$, respectively

Consequently, in the cases under consideration, system (3.1') allows nonzero periodic solutions $z^*(\tau)$. Then $Kz^*(\tau) = x^*(\tau)$ is a periodic solution of the system (3.1).

Thus, the following theorem is proved.

Theorem 3.3. *Under the conditions of the Theorem 3.2, the system (3.1) allowed nonzero multiperiodic solutions enough for the matrix* A *to have at least one eigenvalue* $\lambda = \lambda(A)$ *with the real part* $\operatorname{Re}\lambda(A) = 0$ *equal to zero.*

We have the following theorem from the theorem 3.3, as a corollary.

Theorem 3.4. *Under the conditions of the Theorem 3.3, the system (3.1) did not admit the multiperiodic solution other than trivial, it is sufficient that all eigenvalues of the matrix* A *have nonzero real parts*.

Since the system (3.1) is (θ, ω)-periodic, of particular interest is the question of the existence of its nonzero multiperiodic solutions with the same periods.

The general solution x of the system (3.1) can be represented in the form

$$x(\tau, t, \zeta) = X(\tau)u(\tau, t, \zeta), \tag{3.23}$$

where $u = u(\tau, t, \zeta)$ is the zero of the operator D with the general initial condition for $\tau = 0$

$$x(0, t, \zeta) = u(0, t, \zeta) = u_0(t, \zeta),$$

$X(\tau) = \exp[A\tau]$ is the matricant of the system.

Among the zeros of the operator D there exist multiperiodic ones, in particular, constants by the Theorem 2.1.

Theorem 3.5. *Under the conditions (2.2) - (2.4), the system (3.1) had* (θ, ω) *-periodic with respect to* (τ, t) *solutions of the form (3.23) corresponding to the multiperiodic zero of the operator* D *with the same periods, it is necessary and sufficient that the monodromy matrix* $X(\theta)$ *satisfies condition*

$$\det[X(\theta) - E] = 0. \tag{3.24}$$

Proof. Under the conditions of the theorem, its justice is equivalent to the solvability of equation

$$X(\tau + \theta)u = X(\tau)u \tag{3.25}$$

in the space of (θ, ω) -periodic with respect to (τ, t) zeros $u = u(\tau, t, \zeta)$ of the operator D.

We arrive at the solvability of the system of equations

$$[X(\theta) - E]u = 0,$$

which is equivalent to the condition (3.24) taking into account the properties of the matricant $X(\tau + \theta) = X(\tau)X(\theta)$ from the system (3.25).

In conclusion, we note that the fulfillment of condition

$$\det[X(\theta) - E] \neq 0 \tag{3.26}$$

guarantees the absence of such solutions.

We also note that condition (3.24) is a sufficient sign of the existence of the nonzero multiperiodic solution of the system (3.1).

Theorem 3.6. Let conditions (2.2) - (2.4) and (3.26) be satisfied. Then the system (3.1) allowed nonzero (θ, ω) -periodic solutions of the form (3.23) necessary and sufficient for the functional-difference equations

$$u(\tau+\theta, t+q\omega, \zeta) = [X(\theta) - E]^{-1} X(\theta)[u(\tau+\theta, t+q\omega, \zeta) - u(\tau, t, \zeta)], \ q \in Z^m \tag{3.27}$$

to be solvable in the space of zeros of the operator D.

Proof. Under the condition (3.26) from the definition of (θ, ω) -periodicity with respect to (τ, t) of solution (2.23), we have the equation (3.27). We must be to take into account that $u(\tau, t, \zeta)$ is the zero of the operator D to complete the proof.

If the equation (3.27) has only zero solutions, then, under the condition (3.26), the system (3.1) does not have a nontrivial multiperiodic solution.

We also note that the fulfillment of the condition

$$\operatorname{Re}\lambda_j(A) \neq 0, \ j = \overline{1, n}$$

on the non-zero real parts $\operatorname{Re}\lambda_j(A)$ of all eigenvalues $\lambda_j(A)$ of the matrix A ensures the fulfillment of condition (3.26).

In conclusion, we note that on the basis of the multiperiodic structures (2.30) and (3.13) the characteristics $\mu(\tau^0, \tau, \zeta)$ of the matricant $X(\tau)$ and by the theorems which proved above, it is easy to obtain structures of (θ, ω) -periodic with respect to (τ, t) solutions of the system (3.1) expressed in terms of variables $\tau, \hat{\tau}, s, \sigma, t, \zeta$.

4. The multiperiodic structure of an inhomogeneous linear system with operator D. Consider the inhomogeneous linear equation

$$Dx = Ax + f(\tau, t, \zeta) \tag{4.1}$$

corresponding to the homogeneous equation (3.1), where the n-vector function $f(\tau, t, \zeta)$ satisfies condition

$$f(\tau+\theta, t+q\omega, \zeta) = f(\tau, t, \zeta) \in C^{(0,\hat{e},\tilde{e})}_{\tau,t,\zeta}(R \times R^m \times R^l). \tag{4.2}$$

Assume that the condition (3.26) is fulfilled and we search for the (θ,ω)-periodic with respect to (τ,t) solution $x(\tau,t,\zeta)$ of the system (4.1) that corresponds to zero $u(\tau,t,\zeta)$ of the operator D possessing the property of multiperiodicity with the same periods (θ,ω) for (τ,t).

Therefore, we have the solution

$$x(\tau,t,\zeta)=X(\tau)u(\tau,t,\zeta)+X(\tau)\int_0^\tau X^{-1}(s)f(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))ds \qquad (4.3)$$

with zero $u(\tau+\theta,t+q\omega,\zeta)=u(\tau,t,\zeta),\ q\in Z^m$ of the operator D having the property $x(\tau+\theta,t+q\omega,\zeta)=x(\tau,t,\zeta),\ q\in Z^m$.

Then the solution (4.3) has another representation

$$x(\tau,t,\zeta)=X(\tau+\theta)u(\tau,t,\zeta)+X(\tau+\theta)\int_0^{\tau+\theta} X^{-1}(s)f(s,\lambda(s,\tau+\theta,t),\mu(s,\tau+\theta,\zeta))ds. \qquad (4.4)$$

Further, we obtain

$$x(\tau,t,\zeta)=\left[X^{-1}(\tau+\theta)-X^{-1}(\tau)\right]^{-1}\left[\int_0^{\tau+\theta} X^{-1}(s)f(s,\lambda(s,\tau+\theta,t),\mu(s,\tau+\theta,\zeta))ds+\right.$$

$$\left.+\int_\tau^0 X^{-1}(s)f(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))ds\right],$$

(4.5)

eliminating from identities (4.3) and (4.4) the unknown zero $u(\tau,t,\zeta)$ of the operator D, where the reversible of the matrix $\left[X^{-1}(\tau+\theta)-X^{-1}(\tau)\right]$ follows from condition (3.26).

If we accept the notation

$$f_\theta(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))=\begin{cases} f(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta)),\ \tau\xrightarrow{s}0,\\ f(s,\lambda(s,\tau+\theta,t),\mu(s,\tau+\theta,\zeta)),\ 0\xrightarrow{s}\tau+\theta,\end{cases}$$

then formula (4.5) can be represented in a more compact form

$$x(\tau,t,\zeta)=\left[X^{-1}(\tau+\theta)-X^{-1}(\tau)\right]^{-1}\int_\tau^{\tau+\theta} X^{-1}(s)f_\theta(s,\lambda(s,\tau,t),\mu(s,\tau,\zeta))ds. \qquad (4.6)$$

where $\gamma\xrightarrow{s}\delta$ means changes in the variable s from γ to δ. Obviously, if the system (3.1) does not have multiperiodic solutions, except for zero, then the solution (4.6) of the system (4.1) is a unique multiperiodic solution.

Further, we have solutions

$$\hat{x}(s,\sigma,\hat{\tau},\tau,t,\zeta)=\left[\hat{X}^{-1}(\tau+\theta,\hat{\tau}+\hat{e}\theta)-\hat{X}^{-1}(\tau,\hat{\tau})\right]^{-1}\times$$

$$\times\int_\tau^{\tau+\theta} X^{-1}(\varepsilon)f_\theta(\varepsilon,\lambda(\varepsilon,\tau,t),h(\varepsilon-s,z(\varepsilon),\zeta-z(\sigma)))d\varepsilon \qquad 4.7)$$

of the equation

$$\overline{\overline{D}}\hat{x}=A\hat{x}+f(\tau,t,\zeta)$$

with the differentiation operator (3.18) from representation (4.6) on the basis of multiperiodic structures (2.30) and (3.13) of the quantity $\mu(s,\tau,\zeta)$ and $X(\tau)$.

Thus, the following theorem is proved.

Theorem 4.1. *Assume that conditions (2.2) - (2.4), (3.26) and (4.2) are satisfied, and the homogeneous system (3.1) does not have multiperiodic solutions except zero. Then the system (4.1) has a unique* (θ,ω) *-periodic solution (4.6) for which the* $(\alpha,\beta,\gamma,\theta,\omega)$ *-periodic with respect to* $(s,\sigma,\hat{\tau},\tau,t)$ *structure (4.7) satisfies equation (4.8) with the differentiation operator (3.18).*

In conclusion, note that we can derive the multiperiodic structure of the general solution (4.3) of the system (4.1) similarly to formula (4.7).

Conclusion. A method for studying the multiperiodic structure of oscillatory solutions of perturbed linear autonomous systems of the form (1.1) - (1.2) was developed. The main essence of the method for studying the multiperiodic structures of solution of the system under consideration is a combination of the known methods [1-3] with the methods used in [15, 16] for the autonomous systems. In this case, some system input received perturbations depending on the time variables τ, t. In conclusion, the sufficient conditions for the existence of the multiperiodic solutions of linear systems (1.1) - (1.2) with the differentiation operator D in the directions of a toroidal vector field with respect to time variables and of the form of Lyapunov's systems with respect to space variables were established. Moreover, relation (4.6) is an integral representation of the multiperiodic solution of the system, and (4.7) determines its multiperiodic structure. We also note that the integral representation given here differs from the analogue given in [15, 16].

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ВЕКТОРЛЫҚ ӨРІС БОЙЫНША ДИФФЕРЕНЦИАЛДАУ ОПЕРАТОРЛЫ ҚОЗДЫРЫЛҒАН СЫЗЫҚТЫ АВТОНОМДЫҚ ЖҮЙЕЛЕРДІҢ КӨППЕРИОДТЫ ШЕШІМДЕРІН ЗЕРТТЕУ

Аннотация. Тәуелсіз кеңістік айнымалысына қатысты Ляпунов жүйесі түріндегі және уақыт айнымалысына қатысты көппериодты тороидалды түрдегі векторлық өрістер бағыты бойынша D дифференциалдау операторлы сызықты жүйе қарастырылады. Жүйені анықтайтын барлық берілгендер не уақыт айнымалысынан көппериодты тәуелді, не олардан тәуелсіз болады. Жүйенің автономдық жағдайы бұрынғы жұмыстарда қарастырылған. Бұл жағдайда жүйені анықтайтын кейбір берілгендерге уақыт айнымалысынан тәуелді қоздыртқы берілген. Рационалды өлшенбейтін жиіліктердің жекеленген периодты қозғалыстарының суперпозициясы түріндегі жүйе арқылы сипатталған ізделінді қозғалыс туралы сұрақ зерттеледі. Бастапқы есептер және қозғалыстардың көппериодтылығы туралы есептер зерттеледі. Есептің шешімін анықтау кезінде жүйе бастапқы нүктеден шығатын характеристика маңайында интегралданатыны, одан кейін бастапқы берілгендер характеристикалық жүйенің бірінші интегралдарымен ауыстырылатыны белгілі. Сонымен ізделінді шешім келесі компоненттерден тұрады: D операторының характеристикалық жүйесінің характеристикасы мен бірінші интегралдары, жүйенің бос мүшесі мен матрицанты. Бұл компоненттердің зерттелуші жүйемен сипатталған қозғалыстың көппериодтылық табиғатын ашу кезінде маңызды мағынасы бар болатын периодты және периодты емес құрылымдық құраушылары болады. Шешімді ерекшеленген көппериодты құраушылар арқылы сипаттауды шешімнің көппериодтылық құрылымы деп аталған. Ол көп айнымалылы периоды функциялар мен бір айнымалылы квазипериодты функцияларының байланысы туралы Бордың танымал теоремасы негізінде жүзеге асады. Сонымен, жүйелерді анықтайтын берілгендері қоздырылған жағдайда біртекті және біртексіз жүйелердің жалпы және көппериодты шешімдерінің көппериодты құрылымы нақты зерттелген. Осылайша D операторының нөлдері мен жүйенің матрицанты зерттелген. Біртекті және біртексіз жүйелердің көппериодты шешімдерінің бар болуы және болмауы шарттары тағайындалған.

Түйін сөздер: Көппериодты шешім, автономдық жүйе, дифференциалдау операторы, Ляпунов векторлық өрісі, қоздыртқы.

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ИССЛЕДОВАНИЕ МНОГОПЕРИОДИЧЕСКИХ РЕШЕНИЙ ВОЗМУЩЕННЫХ ЛИНЕЙНЫХ АВТОНОМНЫХ СИСТЕМ С ОПЕРАТОРОМ ДИФФЕРЕНЦИРОВАНИЯ ПО ВЕКТОРНОМУ ПОЛЮ

Аннотация. Рассматривается линейная система с операторомдифференцирования *D* по направлениям векторных полей вида системы Ляпунова относительно пространственных независимых переменных и многопериодического тороидального вида относительно временных переменных. Все входные данные системы либо многопериодично зависят от временных переменных, либо от них не зависят. Автономный случай системы рассмотрен в нашей ранней работе. В данном случае некоторые входные данные получили возмущения, зависящие от временных переменных. Исследуется вопрос о представлении искомого движения, описанного системой в виде суперпозиции отдельных периодических движений рационально несоизмеримых частот. Изучаются начальные задачи и задачи о многопериодичности движений. Известно, что при определении решений задач система интегрируется вдоль характеристик, исходящих из начальных точек, а затем, начальные данные заменяются первыми интегралами характеристических систем. Таким образом, искомое решение состоит из следующих компонентов: характеристик и первых интегралов характеристических систем оператора *D*, матрицанта и свободного члена самой системы. Эти компоненты, в свою очередь, имеют периодические и непериодические структурные составляющие, которые имеют существенное значение при раскрытии многопериодической природы движений, описанных исследуемой системой. Представление решения с выделенными многопериодическими составляющими названо многопериодической структурой решения. Оно реализуется на основе известной теоремы Бора о связи периодической функции от многих переменных и квазипериодической функции одной переменной. Таким образом, более конкретно, исследуются многопериодические структуры общих и многопериодических решений однородных и неоднородных систем с возмущенными входными данными. В таком духе изучаются нули оператора *D* и матрицант системы. Устанавливаются условия отсутствия и существования многопериодических решений как однородных, так и неоднородных систем.

Ключевые слова: Многопериодическое решение, автономная система, оператор дифференцирования, Ляпунова векторное поле, возмущение.

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REFERENCES

[1] Kharasakhal V.Kh. **(1970)** Almost-periodic solutions of ordinary differential equations [Pochti-periodicheskie resheniia obyknovennykh differentsialnykh uravnenii]. Alma-Ata, Nauka (in Russian).

[2] Umbetzhanov D.U. **(1979)** Almostmultiperiodic solutions of partial differential equations [Pochti mnogoperiodicheskie resheniia differentsialnykh uravnenii v chastnykh proizvodnykh]. Alma-Ata: Nauka (In Russian).

[3] Kulzhumieva A.A., Sartabanov Zh.A. **(2013).** Periodic solutions of the systems of differential equations with multidimensional time [Periodicheskie resheniia system differentsialnykh uravnenii s mnogomernym vremenem]. Uralsk: RITs ZKGU (in Russian).ISBN 978-601-266-128-6.

[4] Sartabanov Zh. (1989) Pseudoperiodic solutions of a system of integro differential equations, Ukrainian Mathemstical Journal, 41:1, 116-120. DOI: 10.1007/BF01060661

[5] Berzhanov A.B., Kurmangaliev E.K. **(2009)** Solution of a countable system of quasilinear partial differential equations multiperiodic in a part of variables, Ukrainian Mathematical Journal,61:2, 336-345. DOI:10.1007/s11253-009-0202-4

[6] Kenzhebaev K.K., Abdikalikova G.A., Berzhanov A.B. **(2014)** Multiperiodic Solution of a Boundary-Value Problem for one Class of Parabolic Equations with Multidimensional Time, Ukrainian Mathematical Journal, 66:5, 719-731. DOI:10.1007/s11253-014-0967-y

[7] Abdikalikova G., Berzhanov A. **(2014)** On multiperiodicity and almost periodicity of solutions of boundary value problem for system of parabolic type equation, AIP Conference Proceedings, 1611, 58-62. DOI:10.1063/1.4893804

[8] Muhambetova B.Zh., Sartabanov Zh.A., Kulzhumieva A.A. **(2015)** Multiperiodic solutions of systems of equations with one quasi-linear differential operator in partial derivatives of the first order, Bulletin of the Karaganda University-Mathematics, 78:2, 112-117.

[9] Kulzhumiyeva A.A., Sartabanov Zh.A. **(2016)** On reducibility of linear D-e-system with constant coefficients on the diagonal to D-e-system with Jordan matrix in the case of equivalence of its higher order one equation // Bulletin of the Karaganda University-Mathematics, 84:4, 88-93.

[10] Kulzhumiyeva A.A., Sartabanov Z.A. **(2017)** On multiperiodic integrals of a linear system with the differentiation operator in the direction of the main diagonal in the space of independent variables, Eurasian Mathematical Journal, 8:1, 67-75.

[11] Kulzhumiyeva A.A., Sartabanov Zh.A. **(2017)** Reduction of linear homogeneous D-e-systems to the Jordan canonical form, News of the National Academy of Sciences of the Republic of Kazakhstan. Series Physico-Mathematical, 5:315, 5-12.

[12] Sartabanov Z.A. **(2017)** The multi-period solution of a linear system of equations with the operator of differentiation along the main diagonal of the space of independent variables and delayed arguments, AIP Conference Proceedings, 1880, 040020-1 - 040020-5. DOI: 10.1063/1.5000636

[13] Kulzhumiyeva A.A., Sartabanov Zh.A. **(2018)** General bounded multiperiodic solutions of linear equations with differential operator in the direction of the mail diagonal, Bulletin of the Karaganda University-Mathematics, 92:4, 44-53. DOI:10.31489/2018M4/44-53

[14] Kulzhumiyeva A.A., Sartabanov Z.A. **(2019)** Integration of a linear equation with differential operator, corresponding to the main diagonal in the space of independent variables, and coefficients, constant on the diagonal, Russian Mathematics, 63: 6, 29-41. DOI:10.3103/S1066369X19060045

[15] Sartabanov Z.A.,Omarova B.Z. **(2018)** Multiperiodic solutions of autonomous systems with operator of differentiation on the Lyapunov's vector field, AIP Conference Proceedings, 1997, 020041-1 - 020041-4. DOI:10.1063/1.5049035

[16] Sartabanov Zh.A., Omarova B.Zh. **(2019)** On multi-periodic solutions of quasilinear autonomous systems with operator of differentiation on the Lyapunov's vector field, Bulletin of the Karaganda University-Mathematics, 94:2, 70-83.DOI:10.31489/2019M2/70-83

[17] Kerimbekov A., Abdyldaeva E. **(2016)** On the solvability of a nonlinear tracking problem under boundary control for the elastic oscillations described by fredholmintegro-differential equations, IFIP Advances in Information and Communication Technology, 494, 312-321. DOI: 10.1007/978-3-319-55795-3_29

[18] Bergamasco A.P., Dattori da Silva P.L., Gonzalez R.B. **(2017)** Existence and regularity of periodic solutions to certain first-order partial differential equations, Journal of Fourier Analysis and Applications, 23:1, 65–90. DOI: 10.1007/s00041-016-9463-0

[19] Joyal P. **(1998)** Invariance of Poincaré-Lyapunov polynomials under the group of rotations, Electronic Journal of Differential Equations, 1998, 1–8.

[20] Fonda A., Sfecci A. **(2017)** Multiple periodic solutions of hamiltonian systems confined in a box, Discrete and Continuous Dynamical Systems – Series A, 37:3, 1425-1436. DOI:10.3934/dcds.2017059

[21] Asanova A.T. **(1998)** Almost periodic solution of a semilinear parabolic equation, Differential Equations, 34:12, 1705-1707.

[22] Assanova A.T. **(2017)** Periodic solutions in the plane of systems of second-order hyperbolic equations, Mathematical Notes, 101:1, 39-47. DOI:10.1134/S0001434617010047

[23] Asanova A.T., Dzhumabaev D.S. **(2004)** Periodic solutions of systems of hyperbolic equations bounded on a plane, Ukrainian Mathematical Journal, 56:4, 682-694. DOI:10.1007/s11253-005-0103-0

[24] Assanova A.T., Alikhanova B.Zh., Nazarova K.Zh. Well-posedness of a nonlocal problem with integral conditions for third order system of the partial differential equations. News of the National Academy of Sciences of the Republic of Kazakhstan. Physico-Mathematical Series. Vol. 5, No. 321 **(2018)**, PP. 33-41. https://doi.org/10.32014/2018.2518-1726.5 ISSN 2518-1726 (Online), ISSN 1991-346X (Print)

[25] Dzhumabaev D.S., Bakirova E.A., Kadirbayeva Zh.M. An algorithm for solving a control problem for a differential equation with a parameter.News of the National Academy of Sciences of the Republic of Kazakhstan. Physico-Mathematical Series. Vol.5, No.321 **(2018)**, PP. 25-32. https://doi.org/10.32014/2018.2518-1726.4 ISSN 2518-1726 (Online), ISSN 1991-346X

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ON SOME EFFECTS IN THE STRUCTURE OF WHITE DWARFS

Abstract. We review the effects of general relativity, finite temperatures, nuclear composition and rotation which make a substantial contributionto the structure of white dwarfs.First, the mass-radius, mass-central density relations and mass, density profiles of a white dwarf with total mass1.415 $M_{\odot}$ are constructed both in Newtonian gravity and general relativity, which clearly show that the general relativistic effects are significant for massive white dwarfs close to the Chandrasekhar mass limit, consequently, in strong gravitational fields.Second, hot white dwarfs are studied in the framework of general relativity. Basic parameters of white dwarfs such as the central density, pressure, mass, radius and etc. are calculated. It is shown that the effects of finite temperatures play a key role in low mass white dwarfs.Third, cold white dwarfs are investigated within general relativity employing the Salpeter equation of state. Finally, we investigate the equilibrium configurations of uniformly rotating white dwarfs, using Chandrasekhar and Salpeter equations of state in Newtonian gravity and plot mass-radius, mass-central density relations. It is demonstrated that the effects of rotation are essential in the structure of white dwarfs in allmass range.
Key words: white dwarfs, general relativity, finite temperature, nuclear composition, rotation.

1. Introduction

A white dwarf or degenerate dwarf is the final stage in the evolution of normal (main sequence) stars with masses from 0.08 $M_{\odot}$ to 8 $M_{\odot}$[1, 2, 3] (even to 12 $M_{\odot}$according to some studies [4]), on the other hand, it is one of the classes of compact objects. The lower limit of a main sequence star mass is associated with the impossibility of the occurrence of a thermonuclear helium synthesis reaction.There are two forces which are counterbalanced with each other in the hydrostatic equilibrium configuration of a non-rotating white dwarf: the outward force of interior pressure gradient and the inward force of gravity. In the case of a rotating white dwarf, the centrifugal force is included.

There is no nuclear fusion in the interior of the white dwarf like a normal star. Consequently, it is not the thermal pressure force keeping the white dwarf in hydrostatic equilibrium.The pressure support in the white dwarf is provided by a degenerate electron gas, whereas most of the mass density is due to a nondegenerate gas of ions[3].

The maximum mass of a non-rotating white dwarf cannot exceed the Chandrasekhar mass limit of 1.44$M_{\odot}$beyond which even the degenerate electron gas cannot prevent the white dwarf from gravitational collapse [2] or type Iasupernova explosion[5], whichtakes place as a result of accretion or merger.In turn, type Ia supernova explosion is used as a standard candle to measure intergalactic distances, understand the past and future expansion of the universe and study the nature of dark energy. From this point of view, it is relevant to study properties of white dwarfs and construct their realistic model.

The paper is organized as follows. We introduce the equations of stellar structure in Sec. 2, namely the equation of hydrostatic equilibrium, mass balance equation for static (non-rotating)white dwarfs in Einstein's relativistic theory of gravity and Newtonian gravity, respectively; with the Hartle's approach that describes the equilibrium configuration of rotating white dwarfs. In Sec. 3, we consider some crucial effects such as the effects of general relativity, finite temperatures, nuclear compositions and rotation.The conclusions are given in Sec. 4. The material and results which were considered here can be used in further studies of astrophysics, cosmology and astronomy.

2. Equations of stellar structure

One can derive the Tolman-Oppenheimer-Volkoff (TOV)equation [6] of hydrostatic equilibrium and mass equation for a non-rotating (spherically symmetric configuration) star within the framework of general relativity in the following form

$$\frac{dP(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2}\left(1+\frac{P(r)}{\rho(r)c^2}\right)\left(1+\frac{4\pi r^3 P(r)}{m(r)c^2}\right)\left(1-\frac{2Gm(r)}{rc^2}\right)^{-1}, \quad (1)$$

$$\frac{dm(r)}{dr} = 4\pi r^2\rho(r), \rho(r) = \frac{\varepsilon(r)}{c^2}, \quad (2)$$

where$G$ is the gravitational constant, c is the speed of light and$P(r)$,$m(r)$and$\rho(r)$are the pressure, mass and density profiles, respectively, which depend on the radial coordinate of r.Eqs. (1) and (2) reduces to the expressions

$$\frac{dP(r)}{dr} = -\frac{Gm(r)}{r^2}\rho(r), \quad (3)$$

$$\frac{dm(r)}{dr} = 4\pi r^2\rho(r) \quad (4)$$

in their Newtonian limit.

The equations of stellar structure of slowly and uniformly rotating axially symmetric configurations of white dwarfs can be obtained by using the Hartle's approach in Einstein's relativistic theory of gravity and Newtonian gravity [7-9]. In the Hartle's approach, the axially symmetric configuration is given for a uniform angular velocity sufficiently slow so that the changes in pressure($P_{rot} = P_{st} + \delta P$), energy density($\varepsilon_{rot} = \varepsilon_{st} + \delta\varepsilon$), and gravitational field($\Phi_{rot} = \Phi_{st} + \delta\Phi$)are small. These small changes are considered as perturbations of the known non-rotating solution. The field equations are expanded in powers of the angular velocity and the perturbations are calculated by retaining only the first- and second-order terms[7, 8].

The equation of hydrostatic equilibrium and mass balance equation is supplemented by the equation of state and boundary condition to determine the stellar structure. The equation of state is necessary to describe the properties of the interior matter of a white dwarf. It determines the dependence of the total pressure on the total density for the case of a one-parameter equation of state, $P = P(\rho)$, where P is the pressure and ρ is the density of matter. This form of the equation of state is appropriate when the temperature is a known function of the density inside the star [7]. In this work, we have used the Chandrasekhar equation of state at zero and finite temperatures and the Salpeter equation of state, which takes into account the nuclear composition, electrostatic interaction, Thomas-Fermi correction, and inverse beta decay threshold. Details of these equations of state can be found in Refs. [10-13].

2.Results and Discussions

Effects of general relativity.We solve numerically the equations of stellar structure employing the Chandrasekhar EoS ($\mu = A/Z = 2$)with given boundary conditions and obtain the main parameters of white dwarfs, for instance, mass, radius etc. We also construct the dependence of the mass on the radius in Fig. 1 as well as the dependence of the mass on the central density in Fig. 2.The solid curve indicates mass-radius relation in Newtonian gravity, the dashed curve in general relativity. As it can be seen from

Figure 1, the masses of white dwarfs increase by decreasing their radii, which is the main difference of white dwarfs from the main sequence stars. This feature shows the contribution of relativistic and quantum corrections in the EoS.

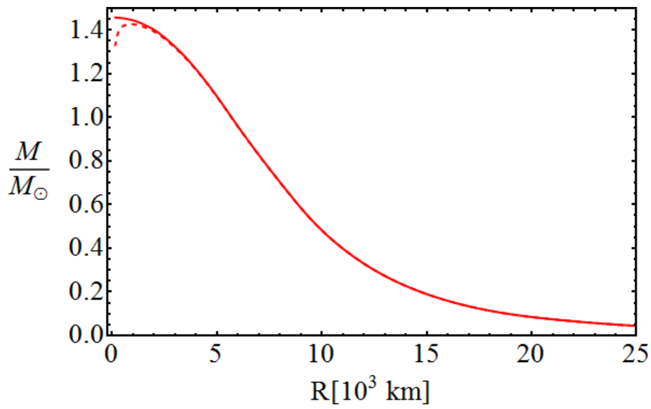


Figure 1 – Mass-radius relations

The mass of a white dwarf also increases by increasing the central density (see Fig. 2). However, it cannot increase infinitely and exceed the Chandrasekhar mass limit $M_{Ch} = 1.44\ M_{\odot}$[5, 6]. The difference between Newtonian and Einstein's gravity is clearly seen for the case of massive white dwarfs. It is related to the presence of general relativistic corrections in the equations of stellar structure.

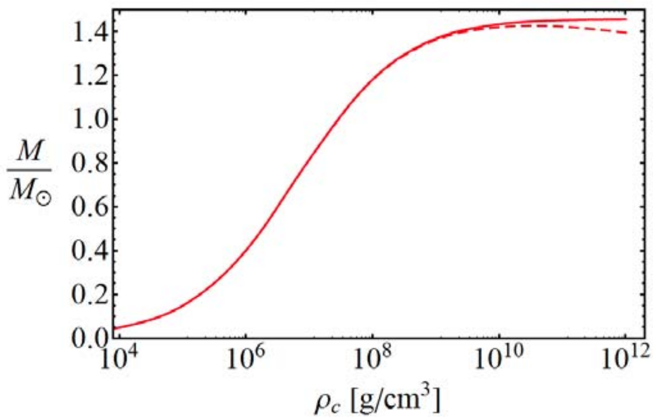


Figure 2 – Mass-central density relations

The growth of mass strengthens gravitational field of white dwarfs. This, in turn, increases pressure, hence in general relativity the maximum mass is less than in Newtonian gravity and it is achieved at finite density. That central density defines stability of white dwarfs in general relativity.

In Newtonian gravity, a white dwarf reaches the maximum mass when the radius tends to zero and the central density and pressure tend to infinity. But, it is impossible, because there is a critical value of central density and central pressure, and consequently, a critical value for the maximum mass. If the value of the central density exceeds this critical value, the white dwarf collapses to a neutron star, or explodes as a type Ia supernova depending on the nuclear composition, temperature etc.The neutronization threshold density is chosen as a critical central density, and the corresponding maximum mass was calculated for white dwarfs composed of ^{12}C in Ref. [14]. The maximum mass M_{max} of a static white dwarf is 1.447 $M_{\odot}$ in Newtonian gravity, and 1.425 $M_{\odot}$ in Einstein's relativistic theory of gravity.

The significance of general relativity for stars can be described by the compactness parameter $z = r_g/R$, where$R$ is the radius of a star, $r_g = 2GM/c^2$ is the gravitational radius (or the Schwarzschild radius), M is the total mass of the star. The compactness parameter of massive white dwarfs close to the

Chandrasekhar mass limit is roughly equal to $z{\sim}0.001$ [14]. One has $z{\sim}0.3$ for neutron stars, $z = 1$ for black holes [15]. That is, the more compactthe object, the more noticeable the role of general relativity [14, 16-18].

We have also reproduced independently the results obtained in the work of Carvalho et al, where they have shown the importance of general relativistic effects for white dwarfs. Following the work [19], in Fig. 3, we show the mass profile of the white dwarf for a fixed total mass $M = 1.415\ \mathrm{M}_{\odot}$, where the importance of general relativistic effects is conspicuous. The total radius of the white dwarf for a fixed total mass$1.415\ \mathrm{M}_{\odot}$ is 938.65 km in general relativity, and 1558.78 km in Newtonian gravity. In Fig. 3 the bluehorizontal dotted line indicates the fixed total mass$M = 1.415\ \mathrm{M}_{\odot}$; the red solid curve indicates the mass profile in Einstein's relativistic theory of gravity; the red dashed curve indicates the mass profile in Newtonian gravity.

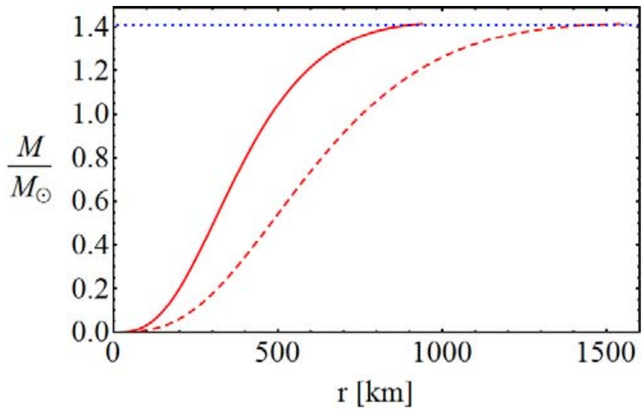


Figure 3 – Mass profiles for a fixed total mass $M = 1.415\ \mathrm{M}_{\odot}$

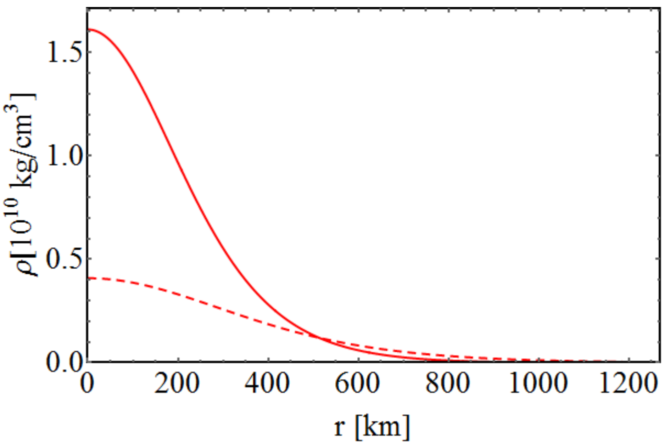


Figure 4 – Density profiles for a fixed mass of$M = 1.415\ \mathrm{M}_{\odot}$

In Fig. 4 we plot the density profile of a white dwarf with fixed mass $M = 1.415\ \mathrm{M}_{\odot}$. The red solid curve denotes the general relativistic density profile; the red dashed curve denotes the Newtonian density profile. From Fig. 4, one can notice that the mass density of the general relativistic white dwarf is larger than the Newtonian one in the central region, where the major part of the white dwarf mass is concentrated. The central density of the white dwarf with a fixed mass $1.415\ M_{\odot}$is $\rho_{cen}^{GR} = 1.61 \times 10^{10} \mathrm{g/cm^3}$ in general relativity, $\rho_{cen}^{NG} = 4.08 \times 10^{9} \mathrm{g/cm^3}$ in Newtonian Gravity [19].

*Effects of finite temperatures.*In Fig. 5, we have constructed the mass-radius relations of general relativistic non-rotating white dwarf cores at finite temperatures$T = (10^4, 10^5, 10^6, 10^7, 4 \times 10^7, 10^8)\ K$ using the Chandrasekhar equation of state $(\mu = 2)$at finite temperatures [20, 21]. From Fig. 5, it can be seen that the effect of finite temperatures increases with decreasing mass and it is significant for low-mass white dwarfs. This indicates that the substance of low-mass white dwarfs cannot be considered completely degenerate.

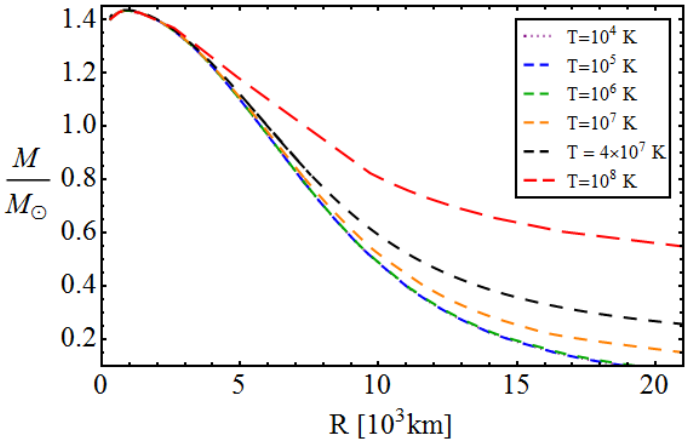
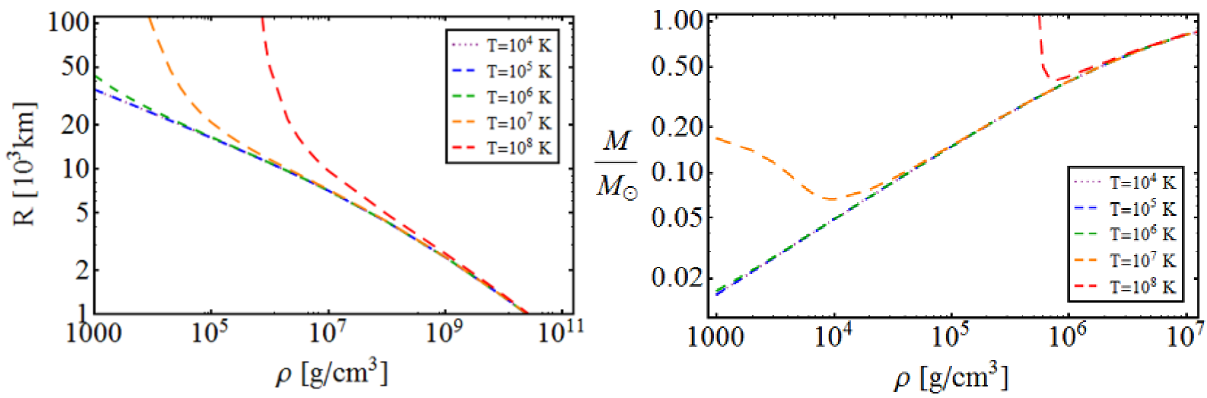


Figure 5 – Mass-radius relations at finite temperatures

Figure 6 shows the radius-central density and mass-central density relations at selected temperatures, where the effects of finite temperatures are more pronounced with a decrease in the central density, and with increasing central density, these effects weaken. That is, the effects of finite temperatures are especially important for white dwarfs with low central densities.



a) b)

Figure 6 – Radius-central density (a) and mass-central density(b) relationsat finite temperatures

*Effects of nuclear composition.*We have considered static white dwarfs by employing the Salpeter equation of state in general relativity and compared them with the results of the Chandrasekhar equation of state.

The Salpeter equation of state allows one to take into account the electrostatic interaction, the Thomas-Fermi correction, and the nuclear composition of white dwarfs. In Figures 7-8, mass-radius, mass-central density and central density-radius ratios were constructed for cold white dwarfs in the general theory of relativity (TOV equation). The plots were constructed for different nuclear compositions of $^{4}_{2}$He, $^{12}_{6}$C, $^{16}_{8}$O, $^{20}_{10}$Ne, $^{24}_{12}$Mg, $^{28}_{14}$Si, $^{56}_{26}$Fe (for the Salpeter EoS) and $\mu = 2$ (for the Chandrasekhar EoS) [22]. The figures show that the heavier the element, the lower the upper limit of the mass of white dwarfs.

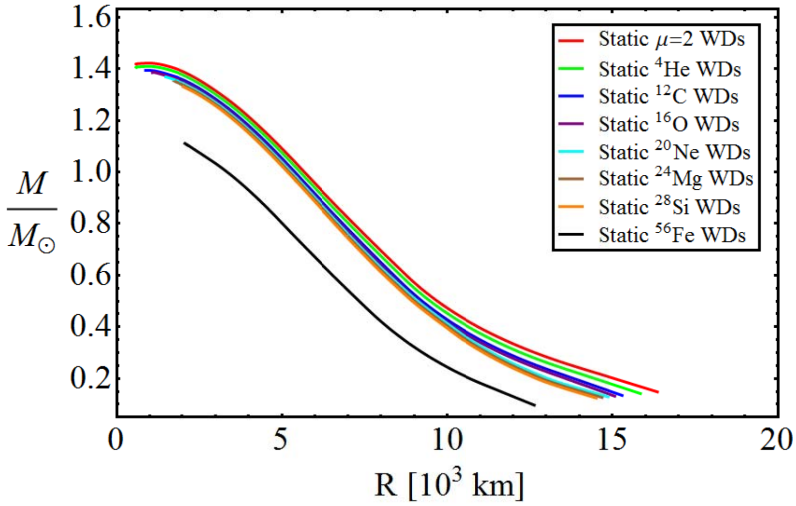
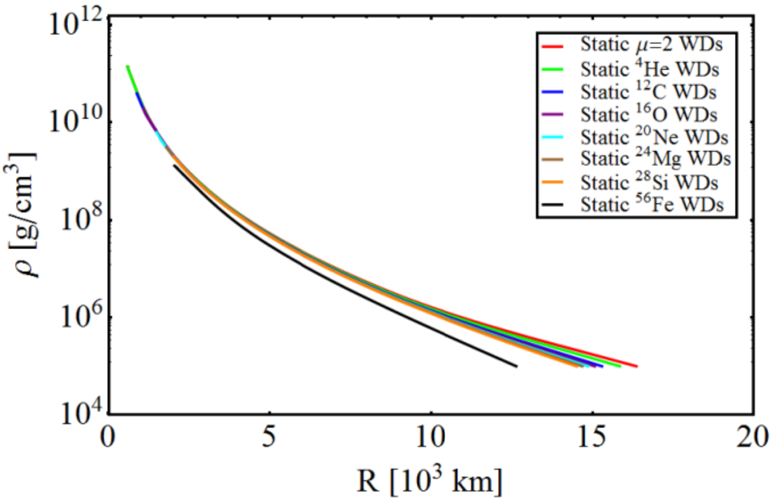
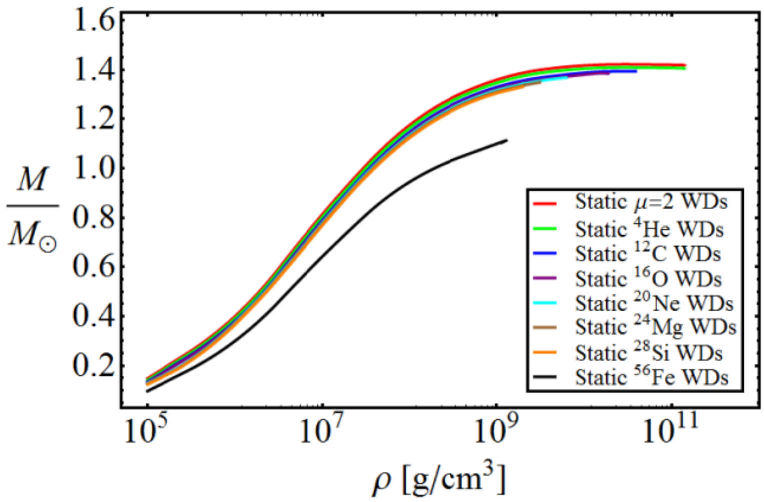


Figure 7 – Mass-radius relations for different nuclear compositions



a)



b)

Figure 8 – Central density-radius (a) and mass-central density (b) relations for different nuclear compositions

In addition, it should be noted that in the Salpeter equation of state, the effect of neutronization (inverse beta decay) for a white dwarf with a uniform nuclear composition was taken into account. This effect sets a limit on the upper limit of the central density($1.37 \times 10^{11} g/cm^3$ for ${}^4_2He$, $3.90 \times 10^{10} g/cm^3$ for ${}^{12}_6C$, $1.90 \times 10^{10} g/cm^3$ for ${}^{16}_8O$, $6.21 \times 10^9 g/cm^3$ for ${}^{20}_{10}Ne$, $3.16 \times 10^9 g/cm^3$ for ${}^{24}_{12}Mg$, $1.97 \times 10^9 g/cm^3$ for ${}^{28}_{14}Si$, $1.14 \times 10^9 g/cm^3$ for ${}^{56}_{26}Fe$) and, therefore, on the mass of white dwarfs[1].

*Effects of rotation.*There are three forces that act on any rotating star: the outward force pressure, the inward force of gravity and the centrifugal force. We follow the Hartle approach [7] and [9, 23], in order to derive the main equations of stellar structure of a rotating star in the case of uniform and slow rotation. The point of Hartle's approach consists in considering a spherically symmetric non-rotating compact object as starting point to construct a rotating star model. The structure equations are obtained up to the second order in the angular velocity$\Omega = \sqrt{GM_{tot}/r_e}$,G is the gravitational constant, M_{tot} is the total of the star, and r_e is the equatorial radius of the star. Afterwards, we calculate the main parameters of a slow and uniform rotating configuration and plot the relations of the main parameters. We show the significance of the rotation effects for the entire range of mass.

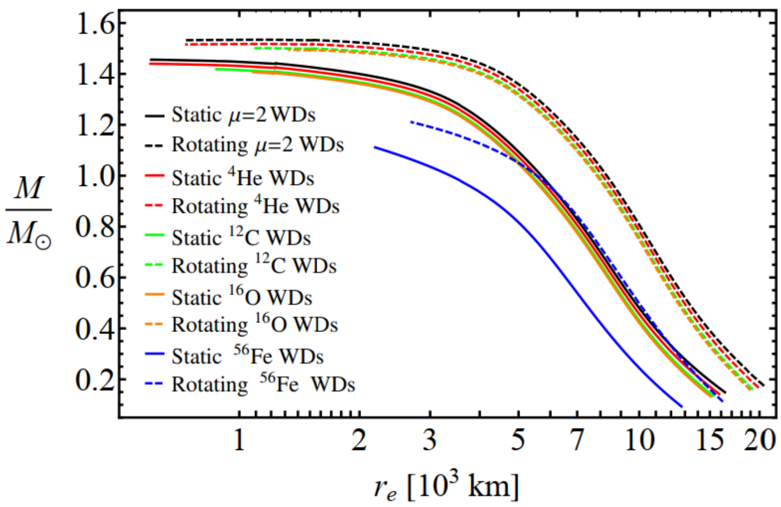


Figure 9 – Mass-equatorial radius relations

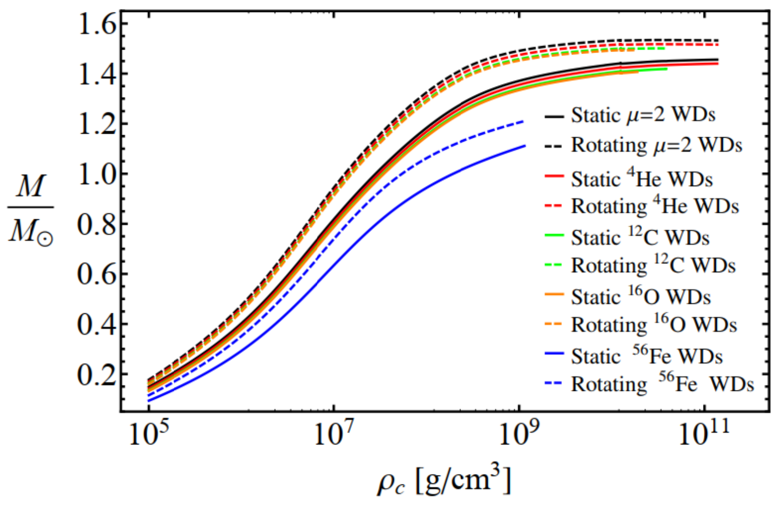


Figure 10 – Mass-central density relations

Fig. 9 shows the mass and equatorial radius relation. We have selected two equations of state: the Chandrasekhar equation of state with average molecular weight μ=2, and the Salpeter equation of state for pure helium ^4He, carbon ^{12}C, oxygen ^{16}O and iron ^{56}Fe white dwarfs, as limiting cases [23]. The equatorial radius for a static case reduces to the static radius. All solid curves indicate non-rotating (static) white dwarfs, whereas all dashed curves indicate rotating white dwarfs at the mass shedding rate. One can see that depending on the equation of state and nuclear composition, white dwarfs display different mass-radius relations.

In Fig. 10, the mass of a white dwarf is shown as a function of the central density. The mass is given in units of one solar mass and the central density is given in g/cm^3. As expected, rotating white dwarfs have larger masses with respect to their static counterparts. In all our computations we restricted the maximum values of the central density to the values of inverse β-decay density to fulfill the stability condition of white dwarfs [17, 24, 25].

5. Conclusions

We considered the structure of non-rotating and slowly rotating, cold and hot, classical and general relativistic white dwarfs in hydrostatic equilibrium. In particular, the mass-radius, mass-central density, radius- central density etc. relations were constructed for white dwarfs using the Chandrasekhar and Salpeter equations of state. We studied the effects of general relativity, finite temperature, nuclear compositions and rotation inthe structure of an equilibrium configuration of white dwarfs. We concluded that:

- the effects of Einstein's general relativity are significant for massive, high density and small radius white dwarfs close to the Chandrasekhar mass limit;
- the finite temperatures considerably affect the structure of white dwarfs at low densities, that is, they play a major role for low-mass white dwarfs;
- the nuclear composition, electrostatic interaction andThomas-Fermi correction are importantfor white dwarfs in all mass range;
- the neutronization threshold is critical near the Chandrasekhar mass limit;
- the uniform rotation are crucial for all white dwarfs in the entire range of mass.

It would be interesting to compare and contrast the observational data for white dwarfs with the theoretical results presented here in analogy to Ref. [26]. In addition, it would be fascinating to investigate the spectral features of white dwarfs in a wide range of X-rays, optical and ultraviolet etc. variabilityfound in symbiotic binary systems in analogy to Ref. [27]. That will be the issue of future studies.

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АҚ ЕРГЕЖЕЙЛІ ЖҰЛДЫЗДАРДЫҢ ҚҰРЫЛЫМЫНДАҒЫ КЕЙБІР ЭФФЕКТТЕР ТУРАЛЫ

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Аннотация. Ақ ергежейлі жұлдыздардың құрылымына үлес қосатын жалпы салыстырмалылық теориясы, шекті температура, ядролық құрам және айналудың әсерлеріқарастырылды. Біріншіден, масса-радиус, масса-орталық тығыздық қатынастары және масса, тығыздық профильдері толық массасы 1.415 $M_\odot$ақ ергежейлі жұлдыз үшін Ньютон гравитациясында, сондай-ақ жалпы салыстырмалылық теориясында тұрғызылды. Олар жалпы салыстырмалылық теориясының әсерлері Чандрасекар массалық шегіне жақын үлкен массадағы ақ ергежейлі жұлдыздар үшін, демек, күшті гравитациялық өрістер үшін маңызды екенін айқын көрсетеді. Екіншіден, ыстық ақ ергежейлі жұлдыздар жалпы салыстырмалылық теориясының шеңберінде зерттеледі. Орталық тығыздық, қысым, масса, радиус және т. б. сияқты ақ ергежейлі жұлдыздардың негізгі параметрлері есептелінді. Массасы аз ақ ергежейлі жұлдыздарда шекті температураның әсерін ескеру қажеттілігі көрсетілді. Үшіншіден, суық ақ ергежейлі жұлдыздар Салпитердің күй теңдеуін қолдана отырып, жалпы салыстырмалылық теориясының шеңберінде зерттеледі. Соңында, Ньютон гравитациясында Чандрасекар мен Салпитердің күй теңдеуін қолдана отырып, бірқалыпты айналатын ақ ергежейлі жұлдыздардың тепе-тең конфигурацияларын зерттелді және масса-радиус, масса-орталық тығыздық қатынастарын тұрғызылды. Айналудың әсерлері массаның барлық мәндерінде ақ ергежейлі жұлдыздардың құрылымы үшін маңызды.

Түйін сөздер: ақ ергежейлі жұлдыздар, жалпы салыстырмалық теориясы, шекті температура, ядролық құрамы, айналу.

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О НЕКОТОРЫХ ЭФФЕКТАХ В СТРУКТУРЕ БЕЛЫХ КАРЛИКОВ

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Аннотация. Рассмотрено влияние общей теории относительности, конечных температур, ядерного состава и вращения, которые вносят существенный вклад в структуру белых карликов. Во-первых, построены соотношения масса-радиус, масса-центральная плотность и профили плотности белого карлика с полной массой 1,415 $M_\odot$ как в ньютоновской гравитации, так и в общей теории относительности, которые четко показывают, что общие релятивистские эффекты важны для массивных белых карликов около предела массы Чандрасекара, следовательно, в сильных гравитационных полях. Во-вторых, горячие белые карлики изучены в рамках общей теории относительности. Вычислены основные параметры белых карликов, такие как центральная плотность, давление, масса, радиус и т. д. Показано, что влияние конечных температур играет ключевую роль в маломассивных белых карликах. В-третьих, холодные белые карлики исследуются в рамках общей теории относительности с использованием уравнения состояния Салпитера. В заключение исследованы равновесные конфигурации равномерно вращающихся белых карликов, используя уравнения состояния Чандрасекара и Салпитера в ньютоновской гравитации и построены зависимости масса-радиус, масса-центральная плотность. Показано, что эффекты вращения необходимы в структуре белых карликов во всем диапазоне масс.

Ключевые слова: белые карлики, общая теория относительности, конечная температура, ядерный состав, вращение.

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REFERENCES

[1] S.L. Shapiro, S.A. Teukolsky,Black holes, white dwarfs, and neutron stars: The physics of compact objects(John Wiley & Sons, New-York, 1983).

[2]I. Sagert, M. Hempel, C. Greiner, J. Schaffner-Bielich,Compact stars for undergraduates, European Journal of Physics, 2006, Vol. 27, No. 3, P. 577-610.

[3] T. Padmanabhan. Theoretical astrophysics, Volume II: Stars and stellar systems (Cambridge University Press, New-York, 2001, P. 575).

[4]G.R. Lauffer. Evolutionary Sequences for H and He Atmosphere Massive White Dwarf Stars, Dissertação (Mestrado) (UniversidadeFederal do Rio Grande do Sul., 2018, P.90).

[5]M.H. van Kerkwijk, P. Chang, S. Justham, Sub-Chandrasekhar white dwarf mergers as the progenitors of type Ia supernovae, Astrophys. J. Lett., 2010, Vol. 722, No.2, P. L157-L161.

[6]R.C. Tolman,Static solutions of Einstein's field equations for spheres of fluid, Phys. Rev., 1939, Vol. 55, No. 4, P. 364-373.

[7]J.B. Hartle, Slowly rotating relativistic stars. I. Equations of structure, Astrophys. J., 1967. , Vol. 150, No. 3, P. 1005-1029.

[8]J.B. Hartle, K.S. Thorne, Slowly rotating relativistic stars. II. Models for neutron stars and supermassive stars, Astrophys. J., 1968, Vol. 153, No. 3, P. 807-834.

[9]K. Boshkayev, H. Quevedo, Z. Kalymova, B. Zhami, Hartle formalism for rotating Newtonian configurations, European Journal of Physics, 2016, Vol. 37, No. 6, id 065602, P. 1-30.

[10] S. Chandrasekhar, The maximum mass of ideal white dwarfs, Astrophys. J., 1931, Vol. 74, P. 81-82.

[11]E.E. Salpeter, Energy and pressure of a zero-temperature plasma, Astrophys. J., 1961, Vol. 134, No. 3, P. 669-682.

[12]E.E. Salpeter, H.M. van Horn, Nuclear reaction rates at high densities,Astrophys. J., 1969, Vol. 155, P. 183-202.

[13]M. Rotondo, J.A. Rueda, R. Ruffini, S. Xue, Relativistic Thomas-Fermi treatment of compressed atoms and compressed nuclear matter cores of stellar dimensions, Phys. Rev. C., 2011, Vol. 83, No. 4, P. 045805-1-045805-15.

[14]K.A. Boshkayev, B.A. Zhami, Z.A. Kalymova, G.S. Balgimbekov, A.S. Taukenova, Z.N. Brisheva, N. Koyshybaev, Theoretical calculation of the main parameters of white dwarfs, News of the National Academy Sciences of the Republic of Kazakhstan,Physico-mathematical series, 2016, Vol. 3, No. 307, P. 49-60.

[15]D.G. Yakovlev, General relativity and neutron stars, Int. J. Mod. Phys.A, 2016, Vol. 31, No. 02n03, P. 1641017-1 - 1641017-10.

[16]M. Rotondo, J.A. Rueda, R. Ruffini, S. Xue,Relativistic Feynman-Metropolis-Teller theory for white dwarfs in general relativity,Phys. Rev. D, 2011, Vol. 84, id 084007, P. 1-16.

[17]K. Boshkayev, J.A. Rueda, R. Ruffini, I. Siutsou,On general relativistic uniformly rotating white dwarfs, Astrophys. J., 2013, Vol. 762, No. 2, id 117, P. 1-14.

[18]K. Boshkayev, J.A. Rueda, R. Ruffini, I. Siutsou, General Relativistic and Newtonian White Dwarfs, Proceedings of the MG13 Meeting on General Relativity (in 3 Volumes). Edited by Rosquist Kjell et al. Published by World Scientific Publishing Co. Pte.Ltd., 2015. ISBN #9789814623995, P. 2468-2474.

[19]G.A. Carvalho, R.M. Marinho Jr., M. Malheiro,General relativistic effects in the structure of massive white dwarfs,Gen. Relativ. Gravit., 2018, Vol. 50, No. 4, id 38, P. 1-16.

[20]K. Boshkayev, J.A. Rueda, B. Zhami, Kalymova Zh., G. Balgymbekov,Equilibrium structure of white dwarfs at finite temperatures, Int. J. Mod. Phys.: Conf. Ser., 2016, Vol. 42, P. 1660129-1 – 1660129-7.

[21]K.A. Boshkayev, B.A. Zhami, Zh.A. Kalymova, Zh.N. Brisheva, Static white dwarfs taking into account the effects of finite temperatures and general relativity,News of the National Academy of Sciences of the Republic of Kazakhstan, Physico-mathematical series, 2017, Vol. 6, No. 316, P. 27-38.

[22]K.A. Boshkayev, B.A. Zhami, Zh.A. Kalymova, Zh.N. Brisheva, A.S. Taukenova, Y.K. Aimuratov,White dwarf stars taking into account nuclear compositionin general relativity,Recent Contributions to Physics, 2018, Vol. 65, No. 2, P. 4-12.

[23]K. Boshkayev, H. Quevedo, B. Zhami, I-Love-Q relations for white dwarf stars, MNRAS, 2017, Vol. 464, No. 4, P. 4349-4359.

[24]M. Rotondo, J.A. Rueda, R. Ruffini, S. Xue,Relativistic Feynman-Metropolis-Teller theory for white dwarfs in general relativity,Phys. Rev. D, 2011, Vol. 84, No. 8, P. 084007-1 – 084007-16.

[25]K. Boshkayev, J.A. Rueda, R. Ruffini,On the maximum mass of general relativistic uniformly rotating white dwarfs,Int. J. Mod. Phys. E, 2011, Vol. 20, P. 136-140.

[26]K. Boshkayev,Equilibrium configurations of rotating white dwarfs at finite temperatures, Astronomy Reports, 2018, Vol. 62, No. 12, P. 847-852.

[27] L.A. Pavlova, Features of X-ray radiation in symbiotic stars, News of the National Academy of Sciences of The Republic of Kazakhstan. Physico-Mathematical Series. 2019, Vol. 3, No. 325, P. 38 - 41. ISSN 1991-346X. https://doi.org/10.32014/2019.2518-1726.22

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SPECTRAL DECOMPOSITION OF A FIRST ORDER FUNCTIONAL DIFFERENTIAL OPERATOR

Abstract. In this paper we study spectral properties of a boundary value problem of a first-order differential equation with constant coefficients and a deviating argument. By spectral properties we mean completeness and basis property of the system of eigenfunctions and associated functions of the boundary value problem, as well as the Volterra property.

Keywords: equation with deviating argument, completeness, basis property, Volterra property, Sturm-Liouville operator, Riesz basis.

1. Introduction

In applications the following eigenvalue problem is often occurred in a more general form [1, p. 520]:

$$Au = \lambda Su .$$

If the operator S is determined by the equality $(Sf)(x) = f(-x)$, let us say, for all $f \in L_2$, then the following generalized spectral problem occurs:

$$Au = \lambda u(-x) .$$

In the case when A is a differential operator, we get a differential equation with deviating argument.

Theory of differential equations with deviating argument is the subject of a huge number of works, among which we note only monographs by A.D. Myshkis [2], L.E. Elsgolts and S.B. Norkin [3], which provides an extensive bibliography. Study of the Sturm - Liouville type boundary value problems for an equation with deviating argument is the subject of a monograph by S.B. Norkin [4]. In this and other works, the spectral questions of differential equations with deviations in higher terms, where there is no spectral parameter, are investigated. Only a few works are devoted to the case when the deviating argument is contained in a term with a spectral parameter. In this regard, we note the works of T.Sh. Kalmenov, S.T. Akhmetova and A.Sh. Shaldanbaev [5], A.M. Ibraimkulov [6], S.T. Akhmetova [7].

Apparently, main theorems of the theory of solvability of differential equations with deviating (delayed) arguments were formulated in the monograph by A.D. Myshkis [2].

T.Sh. Kalmenov [5] laid the foundation to a systematic study of spectral questions for a first-order differential equation with a deviating argument of the indicated form. These ideas were developed in [7].

Apparently, generalized spectral problems of the type

$$Au = \lambda u(-x) ,$$

where A is some differential operator of the first order, were first investigated, at the initiative of T.Sh. Kalmenova, in [5]. In [7], various spectral properties of the generalized spectral problem were studied:

$$y'(x) = \lambda y(1-x), \quad 0 < x < 1,$$

$$\alpha y(0) + \beta y(1) = 0,$$

including conditions of self-adjointness, Volterra property and basis property.

Theorem 1.7.7 of the above paper [7] states the Riesz basis property of the system of root functions of the considered generalized spectral problem for

$$(|\alpha| + |\beta|)(\alpha^4 - \beta^4) \neq 0.$$

Results of this work were transferred to the interval [-1 1], by using another method, in [8], where the final solution to basis property questions of root functions of the generalized spectral problem is given

$$u'(-x) = \lambda u(x), \quad -1 < x < 1,$$

$$u(-1) = \alpha u(1).$$

From results of this work it follows that any such correct boundary-value problem is either Volterra or the system of its root functions forms a Riesz basis.

The method of [5] was generalized in [9], in particular, in this paper two abstract theorems on eigenvalues and root vectors of the operators A and A^2, which may be of independent interest, are proved. We give their statement.

Consider a linear operator A in the Hilbert space H. We suppose that domain $D(A)$ of the operator A is dense in H. Then there exists an operator A^*, which is conjugate to the operator A. Let spectrum of the operator A be discrete. The following proposition holds.

Theorem 1.1. Let number λ_0^2 be an eigenvalue of the operator A^2. If the number λ_0 is not eigenvalue of the operator A, then λ_0 is an eigenvalue of the operator A.

In the next theorem we consider the case of root vectors.

Theorem 1.2. Let λ_0^2 be an eigenvalue of the operator A^2. If λ_0 is not eigenvalue of the operator A, then any root vector u_1 of the operator A^2 (if, ofcourse, it exists), corresponding to the eigenvalue λ_0^2, is an associated element of the operator A, corresponding to the eigenvalue λ_0.

In the work of A.M. Ibraimkulov [6] completeness of the root vectors of a second-order equation was studied. The studies of this author were continued in [10] - [14].

Among recent studies we can note works of W.Watkins [15–16], in which questions on solvability of one-dimensional differential equations with involution were considered, A.P. Khromov and his followers [17-18], which considered solvability of integral and differential equations in partial derivatives with involution.

The variable separation method for solving partial differential equations is based on the spectral theory of one-dimensional differential operators. Spectral theory of self-adjoint and non-self-adjoint ordinary differential operators, which originated in the bowels of mathematical physics equations and began with the classical works of Sturm, Liouville, Steklov and others, has received quite complete development over the past century. The spectral theory of self-adjoint ordinary differential operators is almost complete. In the field of the spectral theory of non-self-adjoint ordinary differential operators, significant results on completeness and basis property of eigenfunctions and associated functions were obtained in the works of M.V. Keldysh [19], V.A. Ilyin [20-25], M. Otelbaev [26], A.A. Shkalikov [27], Radzievsky [28] and many other mathematicians.

Theory of basis property of systems of eigenfunctions and associated functions of non-self-adjoint ordinary differential operators, proposed by V.A. Ilyin, got rapid development. A fairly complete idea about development of the theory of basis property by V.A. Ilyin was given in review articles [29-30].

Compared with the spectral theory of ordinary differential operators, the spectral theory of one-dimensional differential operators with involution is in its infancy. Apparently, the first works on the spectral theory of one-dimensional differential operators with involution were carried out by initiative of T.Sh. Kalmenov [31-35] in the last decade of the present century. These studies were continued in the cycle of works by M.A. Sadybekov and A.M. Sarsenby [36-41]. Over the past decade, researchers' interest in differential equations with deviating arguments has grown markedly, as evidenced by publications [42–58]. Theory of bases is described in detail in [60–63].

In this work, we continue the studies begun in [5].

Problem Formulation. Find a spectral decomposition of the operator

$$Au = u'(1-x),\ \ x \in (0,1), \tag{1}$$

$$D(A) = \{u(x) \in C^1(0,1) \cap C[0,1]: \alpha u(0) + \beta u(1) = 0\}, \tag{2}$$

where α, β are arbitrary complex numbers, satisfying the condition:

$$|\alpha| + |\beta| \neq 0. \tag{3}$$

2. Research Method.

The method is based on the following theorem of N.K. Bari [58].

Theorem. If the sequence $\{\psi_j\}$ is complete in a Hilbert space H, it corresponds to the complete biorthogonal sequence $\{\varphi_j\}$ for any $f \in H$

$$\sum_{j=1}^{\infty}\left|(f, \psi_j)\right|^2 < \infty,\ \ \sum_{j=1}^{\infty}\left|(f, \varphi_j)\right|^2 < \infty, \tag{4}$$

then the system $\{\psi_j\}$ forms a Riesz basis of the Hilbert space H.

Therefore, we first show completeness of the system of eigenfunctions of the operator (1) - (2); then we find complete biorthogonal systems of functions and prove inequalities (4).

2.1. On spectrum of the operator.

Consider the following boundary value problem:

$$y'(1-x) = \lambda y(x); x \in (0,1), \tag{5}$$

$$\alpha y(0) + \beta y(1) = 0 \tag{6}$$

where α, β are arbitrary complex numbers, satisfying the condition (3), and λ is a spectral parameter.

It is easy to note that a general solution of the equation (5) has the form

$$y(x, \lambda) = A\left[\cos\lambda\left(\tfrac{1}{2} - x\right) - \sin\lambda\left(\tfrac{1}{2} - x\right)\right], \tag{7}$$

where $A \neq 0$ is an arbitrary nonzero constant.

Putting (7) into the boundary condition (2), we have

$$\alpha u(0) + \beta u(1) = \alpha A\left(\cos\frac{\lambda}{2} - \sin\frac{\lambda}{2}\right) + \beta A\left(\cos\frac{\lambda}{2} + \sin\frac{\lambda}{2}\right) =$$

$$= A\left[(\alpha+\beta)\cos\frac{\lambda}{2} - (\alpha-\beta)\sin\frac{\lambda}{2}\right] = A^{\circ}\Delta(\lambda) = 0,$$

since $A \neq 0$, therefore

$$\Delta(\lambda) = (\alpha+\beta)\cos\frac{\lambda}{2} - (\alpha-\beta)\sin\frac{\lambda}{2} = 0.$$

Assuming, that $\alpha^2 - \beta^2 \neq 0$, we get

$$\alpha - \beta \neq 0, \qquad \alpha + \beta \neq 0, =>$$

$$tg\frac{\lambda}{2} = \frac{\alpha+\beta}{\alpha-\beta}, \qquad \frac{\lambda_n}{2} = n\pi + arctg\frac{\alpha+\beta}{\alpha-\beta}, =>$$

$$\lambda_n = 2n\pi + 2arctg\frac{\alpha+\beta}{\alpha-\beta}.$$

Lemma 2.1. If

$$\alpha^4 - \beta^4 \neq 0, \tag{8}$$

then the boundary value problem

$$y'(1-x) = \lambda y(x); x \in (0,1), \tag{5}$$

$$\alpha y(0) + \beta y(1) = 0 \tag{6}$$

has infinite set of eigenvalues

$$\lambda_n = 2n\pi + 2arctg\frac{\alpha+\beta}{\alpha-\beta},\ n = 0, \pm1, \pm2, \ldots \tag{9}$$

and the eigenfunctions corresponding to them

$$y_n(x) = A_n\left[\cos\lambda_n\left(\frac{1}{2}-x\right) - \sin\lambda_n\left(\frac{1}{2}-x\right)\right],\ n = 0, \pm1, \pm2, \ldots \tag{10}$$

where A_n are arbitrary constants.

All eigenvalues are simple, i.e. if λ_n is an eigenvalue, then $\Delta(\lambda_n) = 0$ and $\dot{\Delta}(\lambda_n) \neq 0$, where the icon $(\dot{\ })$ means the derivative with respect to the spectral parameter λ.

There are no associated functions.

Proof. From the condition of Lemma it follows that $\alpha + \beta \neq 0,\ \alpha - \beta \neq 0$, then from the equation

$$\Delta(\lambda) = (\alpha+\beta)\cos\frac{\lambda}{2} - (\alpha-\beta)\sin\frac{\lambda}{2} = 0,$$

we have

$$tg\frac{\lambda}{2} = \frac{\alpha+\beta}{\alpha-\beta}.$$

This equation does not have any roots only in two cases:

$$\frac{\alpha+\beta}{\alpha-\beta} = \pm i.$$

This condition holds only when $\alpha = \pm i\beta$, i.e. $\alpha^2 + \beta^2 = 0$, which is possible due to the condition of Lemma 3.1.

In all other cases our equation has roots, which are given by the formulas

$$\lambda_n = 2n\pi + 2arctg\frac{\alpha+\beta}{\alpha-\beta}, \qquad n = 0, \pm1, \pm2, \ldots$$

By (7) we find the corresponding eigenfunctions:

$$y_n(x) = A_n\left[\cos\lambda_n\left(\frac{1}{2}-x\right) - \sin\lambda_n\left(\frac{1}{2}-x\right)\right], \tag{10}$$

where A_n are arbitrary constants.

If λ is a multiple root of the equation $\Delta(\lambda) = 0$, then the system of equations

$$\begin{cases}\Delta(\lambda) = 0,\\ \dot{\Delta}(\lambda) \neq 0\end{cases}$$

implies that $\alpha^2 + \beta^2 = 0$, that is also impossible.

Indeed,

$$\Delta(\lambda) = (\alpha+\beta)\cos\frac{\lambda}{2} - (\alpha-\beta)\sin\frac{\lambda}{2} = 0,$$

$$\dot{\Delta}(\lambda) = \frac{1}{2}\left[-(\alpha+\beta)\sin\frac{\lambda}{2} - (\alpha-\beta)\cos\frac{\lambda}{2}\right] = 0, \Rightarrow$$

$$\begin{cases}(\alpha+\beta)\cos\frac{\lambda}{2} - (\alpha-\beta)\sin\frac{\lambda}{2} = 0,\\ (\alpha-\beta)\cos\frac{\lambda}{2} + (\alpha+\beta)\sin\frac{\lambda}{2} = 0.\end{cases}$$

Since this system of equations has a nontrivial solution, then its determinant will vanish, i.e.

$$\Delta=\begin{vmatrix}\alpha+\beta & -(\alpha-\beta)\\ \alpha-\beta & \alpha+\beta\end{vmatrix}=(\alpha+\beta)^2+(\alpha-\beta)^2=2(\alpha^2+\beta^2)=0.$$

Definition 2.1. If $u(x)$ is an eigenfunction of the boundary value problem

$$Au=u'(1-x)=\lambda u(x); x\in(0,1),$$
$$\alpha u(0)+\beta u(1)=0,$$

then solution of the boundary value problem

$$Bv=v'(1-x)-\lambda v(x)=u(x),$$
$$\alpha v(0)+\beta v(1)=0$$

is called associated function of this boundary value problem.

Now we show that if the condition $\alpha^4-\beta^4\neq 0$ holds, our boundary value problem (5) - (6) does not have associated functions.

Let $u(x)$ be an eigenfunction of the boundary value problem (5) - (6), and $v(x)$ is its corresponding associated function. Then differentiating (5) - (6) by the spectral parameter λ, we have

$$\dot{u}'(1-x)=\lambda\dot{u}+u(x),\quad \alpha\dot{u}(0)+\beta\dot{u}(1)=0,\qquad =>$$
$$\dot{u}'(1-x)-\lambda\dot{u}=u(x),\quad \alpha\dot{u}(0)+\beta\dot{u}(1)=0.$$

Consequently, the difference $z(x)=\dot{u}(x)-v(x)$ is an eigenfunction of our boundary value problem (5) - (6). Then, obviously, the function

$$\dot{u}(x)=v(x)+z(x)$$

is an associated function of our boundary value problem. We prove that it is not possible. Indeed,

$$\alpha u(0)+\beta u(1)=0=\left|u(x)=A\left[\cos\lambda\left(\frac{1}{2}-x\right)-\sin\lambda\left(\frac{1}{2}-x\right)\right]\right|=$$
$$=\alpha A\left(\cos\frac{\lambda}{2}-\sin\frac{\lambda}{2}\right)+\beta A\left(\cos\frac{\lambda}{2}-\sin\frac{\lambda}{2}\right)=$$
$$=A\left[(\alpha+\beta)\cos\frac{\lambda}{2}-(\alpha-\beta)\sin\frac{\lambda}{2}\right]=A\cdot\Delta(\lambda).$$

Differentiating this formula by the spectral parameter λ, we get

$$\frac{d}{d\lambda}[\alpha u(0)+\beta u(1)]=A\cdot\dot{\Delta}(\lambda),$$
$$\alpha\dot{u}(0)+\beta\dot{u}(1)=A\cdot\dot{\Delta}(\lambda)\neq 0,$$

where λ is an eigenvalue of the boundary value problem (5) - (6).

2.2. On completeness.

Lemma 2.2. If

$$\alpha^4-\beta^4\neq 0, \tag{8}$$

then eigenfunctions of the boundary value problem

$$y'(x)=\lambda y(1-x); x\in(0,1), \tag{5}$$
$$\alpha y(0)+\beta y(1)=0 \tag{6}$$

form a complete system in the space $L^2(0,1)$.

Proof. Let $\{y_n\}, n=0,\pm1,\pm2,...$ be a system of eigenfunctions of the boundary value problem (5) - (6). We assume that

$$\int_0^1\overline{f(x)}y_n(x)dx=0,\qquad \int_0^1\overline{f(x)}y_{-n}(x)dx=0, n=0,\pm1,\pm2,...$$

then

$$\int_0^1\overline{f(x)}\left[\cos(2n\pi+2\varphi)\left(\frac{1}{2}-x\right)-\sin(2n\pi+2\varphi)\left(\frac{1}{2}-x\right)\right]dx=0,$$

$$\int_0^1 \overline{f(x)}\left[\cos(2n\pi - 2\varphi)\left(\frac{1}{2} - x\right) + \sin(2n\pi - 2\varphi)\left(\frac{1}{2} - x\right)\right] dx = 0,$$

where

$$\varphi = arctg\frac{\alpha + \beta}{\alpha - \beta}.$$

Supposing $t = \frac{1}{2} - x$, from the first formula we have

$$x = \frac{1}{2} - t, \qquad dx = -dt, =>$$

$$\int_{-1/2}^{1/2} \bar{f}\left(\frac{1}{2} - t\right)[\cos(2n\pi + 2\varphi)t - \sin(2n\pi + 2\varphi)\, t](-dt) =$$

$$= \int_{-1/2}^{1/2} \bar{f}\left(\frac{1}{2} - t\right)[\cos(2n\pi + 2\varphi)t - \sin(2n\pi + 2\varphi)\, t]dt = 0; \qquad (12)$$

Similarly, we obtain

$$= \int_{-1/2}^{1/2} \bar{f}\left(\frac{1}{2} - t\right)[\cos(2n\pi - 2\varphi)t + \sin(2n\pi - 2\varphi)\, t]dt = 0. \qquad (13)$$

Summing up equalities (12) and (13), we get

$$\int_{-1/2}^{1/2} \bar{f}\left(\frac{1}{2} - t\right)[2\cos 2n\pi t \cos 2\varphi t - 2\cos 2n\pi t \sin 2\varphi t]dt = 0,$$

$$\int_{-1/2}^{1/2} \bar{f}\left(\frac{1}{2} - t\right)(\cos 2\varphi t - \sin 2\varphi t)\cos 2n\pi t\, dt = 0, n = 0,1,2,\ldots$$

In this formula supposing $2\pi t = x$, we have

$$t = \frac{x}{2\pi}, \qquad dt = \frac{dx}{2\pi},$$

$$\int_{-\pi}^{+\pi} \bar{f}\left(\frac{1}{2} - \frac{x}{2\pi}\right)\left(\cos\frac{\varphi x}{\pi} - \sin\frac{\varphi x}{\pi}\right)\cos nx dx = 0, \ n = 0,1,2,\ldots \qquad (14)$$

Further, subtracting the formula (13) from (12), we get

$$\int_{-1/2}^{1/2} \bar{f}\left(\frac{1}{2} - t\right)[-2\sin 2n\pi \cdot \sin 2\varphi t - 2\sin 2n\pi t \cdot \cos 2\varphi t]dt = 0, =>$$

$$\int_{-1/2}^{1/2} \bar{f}\left(\frac{1}{2} - t\right)[\cos 2\varphi t + \sin 2\varphi t]\sin 2n\pi t\, dt = 0, \qquad n = 1,2,\ldots \qquad (15)$$

Now from (14) we have

$$\int_{-\pi}^{+\pi} \bar{f}\left(\frac{1}{2} - \frac{x}{2\pi}\right)\left(\cos\frac{\varphi x}{\pi} - \sin\frac{\varphi x}{\pi}\right)\cos nx dx = 0. =>$$

In this formula assuming , we obtain

$$-x = \xi, dx = -d\xi, =>$$

$$-\int_{-\pi}^{+\pi} \bar{f}\left(\frac{1}{2}+\frac{\xi}{2\pi}\right)\left(\cos\frac{\varphi\xi}{\pi}+\sin\frac{\varphi\xi}{\pi}\right)\cos n\xi d\xi = 0, =>$$

$$\int_{-\pi}^{+\pi} \bar{f}\left(\frac{1}{2}+\frac{\xi}{2\pi}\right)\left(\cos\frac{\varphi\xi}{\pi}+\sin\frac{\varphi\xi}{\pi}\right)\cos n\xi d\xi = 0,$$

or denoting ξ by x, we get

$$\int_{-\pi}^{+\pi} \bar{f}\left(\frac{1}{2}+\frac{\varphi}{2\pi}\right)\left(\cos\frac{\varphi x}{\pi}+\sin\frac{\varphi x}{\pi}\right)\cos nxdx = 0,\ n = 0,1,2,\dots \tag{16}$$

Adding (14) to (16), we receive

$$\int_{-\pi}^{+\pi}\left[\frac{\bar{f}\left(\frac{1}{2}-\frac{x}{2\pi}\right)+\bar{f}\left(\frac{1}{2}+\frac{x}{2\pi}\right)}{2}\cos\frac{\varphi x}{\pi}+\frac{\bar{f}\left(\frac{1}{2}+\frac{x}{2\pi}\right)-\bar{f}\left(\frac{1}{2}-\frac{x}{2\pi}\right)}{2}\sin\frac{\varphi x}{\pi}\right]\cdot$$

$$\cdot\cos nxdx = 0. \tag{17}$$

The function in the integral in (17) is even, therefore

$$\int_{0}^{\pi}\left(P\bar{f}\cos\frac{\varphi x}{\pi}+Q\bar{f}\sin\frac{\varphi x}{\pi}\right)\cos nxdx = 0, \qquad n = 0,1,2,\dots,$$

where

$$P\bar{f}(x) = \frac{\bar{f}\left(\frac{1}{2}-\frac{x}{2\pi}\right)+\bar{f}\left(\frac{1}{2}+\frac{x}{2\pi}\right)}{2}, Q\bar{f}(x) = \frac{\bar{f}\left(\frac{1}{2}+\frac{x}{2\pi}\right)-\bar{f}\left(\frac{1}{2}-\frac{x}{2\pi}\right)}{2}.$$

Due to completeness of the system of functions $\{\cos nx\}, n = 0,1,2,\dots$ in the space $L^2(0,\pi)$, we have

$$P\bar{f}\cos\frac{\varphi x}{\pi}+Q\bar{f}\sin\frac{\varphi x}{\pi} = 0$$

Now we transform the formula (15). Assuming $t = -x$, we get

$$\int_{-1/2}^{1/2}\bar{f}\left(\frac{1}{2}+x\right)(\cos 2\varphi x-\sin 2\varphi x)(-dx) =$$

$$\int_{-1/2}^{1/2}\bar{f}\left(\frac{1}{2}+x\right)(\cos 2\varphi x-\sin 2\varphi x)dx = 0.$$

Consequently, we have a pair of the formulas

$$\int_{-1/2}^{1/2}\bar{f}\left(\frac{1}{2}-x\right)(\cos 2\varphi x+\sin 2\varphi x)\sin 2n\pi xdx = 0, \tag{18}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\bar{f}\left(\frac{1}{2}+x\right)(\cos 2\varphi x-\sin 2\varphi x)\sin 2n\pi xdx = 0. \tag{19}$$

Subtracting the formula (18) from the formula (19), then dividing the result by 2, we obtain

$$\int_{-1/2}^{1/2}\left[\frac{\bar f\left(\frac{1}{2}+x\right)-\bar f\left(\frac{1}{2}-x\right)}{2}\cos 2\varphi x-\frac{\bar f\left(\frac{1}{2}+x\right)+\bar f\left(\frac{1}{2}-x\right)}{2}\sin 2\varphi x\right]\cdot$$

$$\cdot \sin 2n\pi x dx = 0,\ n = 1,2,\ldots. \tag{20}$$

The function in the integral in (20) is even, thus

$$\int_0^{1/2}\left(Q\bar f\cos 2\varphi x - P\bar f\sin 2\varphi x\right)\sin 2n\pi x dx = 0, \tag{21}$$

where

$$Q\bar f(x)=\frac{\bar f\left(\frac{1}{2}+x\right)-\bar f\left(\frac{1}{2}-x\right)}{2}, P\bar f(x)=\frac{\bar f\left(\frac{1}{2}+x\right)+\bar f\left(\frac{1}{2}-x\right)}{2}.$$

Assuming $t = 2\pi x$, we change the variable in the formula (21), then

$$x=\frac{t}{2\pi},\qquad dx=\frac{dt}{2\pi},$$

$$\int_0^{\pi}\left[\frac{\bar f\left(\frac{1}{2}+\frac{t}{2\pi}\right)-\bar f\left(\frac{1}{2}-\frac{t}{2\pi}\right)}{2}\cos\frac{\varphi t}{\pi}-\frac{\bar f\left(\frac{1}{2}+\frac{t}{2\pi}\right)+\bar f\left(\frac{1}{2}-\frac{t}{2\pi}\right)}{2}\sin\frac{\varphi t}{\pi}\right]\cdot$$

$$\cdot \sin ntdt = 0,\qquad n = 1,2,\ldots.$$

Due to completeness of the system of functions in the space $L^2(0,\pi)$, we have

$$\frac{\bar f\left(\frac{1}{2}+\frac{x}{2\pi}\right)-\bar f\left(\frac{1}{2}-\frac{x}{2\pi}\right)}{2}\cos\frac{\varphi x}{\pi}-\frac{\bar f\left(\frac{1}{2}+\frac{x}{2\pi}\right)+\bar f\left(\frac{1}{2}-\frac{x}{2\pi}\right)}{2}\sin\frac{\varphi x}{\pi}=0.$$

Consequently,

$$\begin{cases}\dfrac{\bar f\left(\frac{1}{2}-\frac{x}{2\pi}\right)+\bar f\left(\frac{1}{2}+\frac{x}{2\pi}\right)}{2}\cos\dfrac{\varphi x}{\pi}+\dfrac{\bar f\left(\frac{1}{2}+\frac{x}{2\pi}\right)-\bar f\left(\frac{1}{2}-\frac{x}{2\pi}\right)}{2}\sin\dfrac{\varphi x}{\pi}=0,\\ \dfrac{\bar f\left(\frac{1}{2}+\frac{x}{2\pi}\right)+\bar f\left(\frac{1}{2}-\frac{x}{2\pi}\right)}{2}\sin\dfrac{\varphi x}{\pi}-\dfrac{\bar f\left(\frac{1}{2}+\frac{x}{2\pi}\right)-\bar f\left(\frac{1}{2}-\frac{x}{2\pi}\right)}{2}\cos\dfrac{\varphi x}{\pi}=0.\end{cases}$$

Since the determinant of this system

$$\Delta=\begin{vmatrix}\cos\dfrac{\varphi x}{\pi} & \sin\dfrac{\varphi x}{\pi}\\ \sin\dfrac{\varphi x}{\pi} & -\cos\dfrac{\varphi x}{\pi}\end{vmatrix}=-\cos^2\frac{\varphi x}{\pi}-\sin^2\frac{\varphi x}{\pi}=-1\neq 0,$$

then

$$\frac{\bar f\left(\frac{1}{2}-\frac{x}{2\pi}\right)+\bar f\left(\frac{1}{2}+\frac{x}{2\pi}\right)}{2}=0,\qquad \frac{\bar f\left(\frac{1}{2}+\frac{x}{2\pi}\right)-\bar f\left(\frac{1}{2}-\frac{x}{2\pi}\right)}{2}=0.$$

Summing up these formulas, we get $\bar f\left(\frac{1}{2}+\frac{x}{2\pi}\right)=0$ almost for all $x\in[0,\pi]$, consequently, almost for all $x\in[0,1]$ we have the equality $\bar f = 0$, that is required to prove.

2.3. On the conjugate boundary value problem.

We find conjugate boundary value problem to our boundary value problem

$$Ay = y'(1-x) = \lambda y(x), x\in(0,1); \tag{5}$$

$$\alpha y(0)+\beta y(1)=0. \tag{6}$$

Let $z(x)\in D(L^+)$, i.e. belong to the domain of the conjugate problem, then we have the formula

$$(Ay,z)=(y,A^+z),\qquad \forall y\in D(A),\qquad z\in D(A^+).$$

Expanding this formula and integrating by parts, we find $D(A^{+})$.

$$(Ay,z)=\int_0^1 Sy'(x)\overline{z(x)}\,dx=(Sy',z)=(y',Sz)=|Sz(x)=z(1-x)|=$$

$$=\int_0^1 \overline{Sz}\,dy=Sz\cdot y|_0^1-\int_0^1(\overline{Sz})'y(x)dx=\overline{z(1-x)}\cdot y(x)\Big|_0^1-\int_0^1(\overline{Sz})'y(x)dx=$$

$$=\overline{z(0)}y(1)-\overline{z(1)}y(0)+\int_0^1\overline{z'(1-x)}y(x)dx.$$

Equating to zero, outside the integral term, we compose a system of equations:

$$\begin{cases}\alpha y(0)+\beta y(1)=0\\ \overline{z(1)}y(0)-\overline{z(0)}y(1)=0.\end{cases}$$

Since the system of equations has a non-trivial solution, then the determinant vanishes, i.e.

$$-\alpha\bar{z}(0)-\beta\bar{z}(1)=0,\Rightarrow \bar{z}(0)+\beta\bar{z}(1)=0,\Rightarrow \bar{\alpha}z(0)+\bar{\beta}z(1)=0.$$

From the equality

$$(y,A^{+}z)=\int_0^1 y(x)\,\overline{z'(1-x}dx$$

we have

$$A^{+}z=z'(1-x)$$

consequently, conjugate boundary value problem has the following form:

$$A^{+}z=z'(1-x)=\mu z(x),\ x\in(0,1); \qquad (5)^{+}$$

$$\bar{\alpha}z(0)+\bar{\beta}z(1)=0. \qquad (6)^{+}$$

It is easy to note that this problem similar to the boundary value problem (5) - (6). Since $(\bar{\alpha})^4-(\bar{\beta})^4=0\Leftrightarrow\alpha^4-\beta^4=0$, then the conditions on its solvability are also similar. In particularly, Lemma 2.1 yields the following Lemma 2.3.

Lemma 2.3. If

$$(\bar{\alpha})^4-(\beta)^4\neq 0 \qquad (8)^{+}$$

then the boundary value problem $(5)^{+}$ - $(6)^{+}$ has infinite set of eigenvalues:

$$\mu_m=2m\pi+2arctg\frac{\bar{\alpha}+\bar{\beta}}{\bar{\alpha}-\bar{\beta}},\ m=0,\pm1,\pm2,... \qquad (9)^{+}$$

and their corresponding eigenfunctions:

$$z_m(x)=B_m\left[\cos\mu_m\left(\frac{1}{2}-x\right)-\sin\mu_m\left(\frac{1}{2}-x\right)\right]\ \ m=0,\pm1,\pm2,... \qquad (10)^{+}$$

where B_m are arbitrary constants.

All eigenvalues μ_m are simple.

There are no associated functions.

Lemma 2.4. If

$$(\bar{\alpha})^4-(\bar{\beta})^4\neq 0 \qquad (8)^{+}$$

then eigenfunctions $\{z_n\}$ of the boundary value problem

$$A^{+}z=z'(1-x)=\mu z(x),\ x\in(0,1); \qquad (5)^{+}$$

$$\bar{\alpha}z(0)+\bar{\beta}z(1)=0 \qquad (6)^{+}$$

form a complete system in the space $L^2(0,1)$.

2.4. On the biorthogonal system

Lemma 2.5. If the functions

$$y_n(x) = A_n \left[\cos\lambda_n\left(\frac{1}{2} - x\right) - \sin\lambda_n\left(\frac{1}{2} - x\right)\right], \qquad n = 0, \pm1, \pm2, \dots$$

are eigenfunctions of the boundary value problem

$$Ay = y'(1 - x) = \lambda y(x),\ x \in (0,1); \tag{5}$$

$$\alpha y(0) + \beta y(1) = 0, \tag{6}$$

then the functions

$$z_n(x) = \frac{1}{\bar{A}_n}\left[\cos\bar{\lambda}_n\left(\frac{1}{2} - x\right) - \sin\bar{\lambda}_n\left(\frac{1}{2} - x\right)\right]$$

are eigenfunctions of the conjugate boundary value problem

$$A^{+}z = z'(1 - x) = \mu z(x),\ x \in (0,1); \qquad (5)^{+}$$

$$\bar{\alpha} z(0) + \bar{\beta} z(1) = 0; \qquad (6)^{+}$$

moreover, we have the formula

$$(y_n, z_m) = \delta_{nm},$$

where δ_{nm} is the Kronecker symbol.

Proof.

$$(y_n, z_m) =$$

$$= \int_0^1 \left[\cos\lambda_n\left(\frac{1}{2} - x\right) - \sin\lambda_n\left(\frac{1}{2} - x\right)\right] \cdot \overline{\left[\cos\bar{\lambda}_m\left(\frac{1}{2} - x\right) - \sin\bar{\lambda}_m\left(\frac{1}{2} - x\right)\right]}\, dx$$

$$= \int_0^1 \left[\cos\lambda_n\left(\frac{1}{2} - x\right) - \sin\lambda_n\left(\frac{1}{2} - x\right)\right]\left[\cos\lambda_m\left(\frac{1}{2} - x\right) - \sin\lambda_m\left(\frac{1}{2} - x\right)\right] dx =$$

$$= \int_0^1 \left\{\left[\cos\lambda_n\left(\frac{1}{2} - x\right)\cos\lambda_m\left(\frac{1}{2} - x\right) + \sin\lambda_n\left(\frac{1}{2} - x\right)\sin\lambda_m\left(\frac{1}{2} - x\right)\right] - \right.$$

$$- \left[\cos\lambda_n\left(\frac{1}{2} - x\right)\sin\lambda_m\left(\frac{1}{2} - x\right)\right] - \left[\cos\lambda_n\left(\frac{1}{2} - x\right)\sin\lambda_m\left(\frac{1}{2} - x\right) + \right.$$

$$\left. \left. + \sin\lambda_n\left(\frac{1}{2} - x\right)\cos\lambda_m\left(\frac{1}{2} - x\right)\right]\right\} dx =$$

$$= \int_0^1 \left[\cos(\lambda_n - \lambda_m)\left(\frac{1}{2} - x\right) - \sin(\lambda_n + \lambda_m)\left(\frac{1}{2} - x\right)\right] dx =$$

$$= \left[-\frac{\sin(\lambda_n - \lambda_m)\left(\frac{1}{2} - x\right)}{\lambda_n - \lambda_m}\Bigg|_0^1 - \frac{\cos(\lambda_n + \lambda_m)\left(\frac{1}{2} - x\right)}{\lambda_n + \lambda_m}\Bigg|_0^1\right] =$$

$$= \frac{2\sin\frac{\lambda_n - \lambda_m}{2}}{\lambda_n - \lambda_m} = \frac{\sin\frac{\lambda_n - \lambda_m}{2}}{\frac{\lambda_n - \lambda_m}{2}} = \left|\begin{matrix} \frac{\lambda_n}{2} = n\pi + arctg\,\frac{\alpha + \beta}{\alpha - \beta} \\ \frac{\lambda_n - \lambda_m}{2} = (n - m)\pi \end{matrix}\right| =$$

$$= \frac{\sin(n - m\pi)}{(n - m)\pi} = 0,$$

when $n \neq m$.

If $n = m$, then

$$(y_n, y_m) = \int_0^1 \left[\cos\lambda_n\left(\frac{1}{2}-x\right) - \sin\lambda_n\left(\frac{1}{2}-x\right)\right]^2 dx =$$

$$= \int_0^1 \left[1 - 2\sin\lambda_n\left(\frac{1}{2}-x\right)\cos\lambda_n\left(\frac{1}{2}-x\right)\right] dx =$$

$$= \int_0^1 [1 - \sin\lambda_n(1-2x)]dx = \left[x - \frac{\cos\lambda_n(1-2x)}{2\lambda_n}\right]\Bigg|_0^1 = 1$$

2.5. On basis property

Definition 2.1. Sequence $\{\varphi_j\}$ of vectors of the Banach space B is called basis of this space, if each vector $x \in B$ is expended uniquely in a series

$$x = \sum_{j=1}^{\infty} c_j \times \psi_j$$

converging by the norm of the space B.

Any bounded invertible operator transforms any orthonormal basis into some other basis of the space H. The basis $\{\psi_j\}$ of the Hilbert space H, obtained from the orthonormal basis by using such transformation, is called basis equivalent to orthonormal or the Riesz basis.

Theorem (N.K. Bari). If the sequence $\{\psi_j\}$ is complete in the Hilbert space H, it corresponds to a complete biorthogonal sequence $\{\psi_j\}$ and for all $f \in H$

$$\sum_{j=1}^{\infty}\left|(f,\psi_j)\right|^2 < \infty,\ \sum_{j=1}^{\infty}\left|(f,\varphi_j)\right|^2 < \infty \tag{4}$$

then the system $\{\psi_j\}$ forms Riesz basis in the space H.

Using this theorem, we prove basis property of the system of eigenfunctions of our boundary value problem

$$Ay = y'(1-x) = \lambda y(x),\ x \in (0,1); \tag{5}$$

$$\alpha y(0) + \beta y(1) = 0, \tag{6}$$

where α, β are arbitrary constants, satisfying the condition

$$\alpha^4 - \beta^4 \neq 0. \tag{8}$$

and λ is a spectral parameter.

Our sequence is complete (see Lemma 2.2) in the space $H = L^2(0,1)$, and it corresponds to a biorthogonal sequence $\{z_n\}$ (see Lemma 2.5), which is also complete in H (see Lemma 2.4), therefore, it only remains for us to prove inequalities (4):

Let

$$a_n = (f, y_n) = \int_0^1 f \cdot \bar{y}_n(x)dx,\ b_n = (f, z_n) = \int_0^1 f(x)\bar{z}_n(x)dx. \tag{22}$$

In our case

$$y_n(x) = (-1)^n \left[\cos\lambda_n\left(\frac{1}{2}-x\right) - \sin\lambda_n\left(\frac{1}{2}-x\right)\right],$$

$$z_n(x) = (-1)^n \left[\cos\bar{\lambda}_n\left(\frac{1}{2}-x\right) - \sin\bar{\lambda}_n\left(\frac{1}{2}-x\right)\right].$$

First we transform the integrals (6).

$$a_n = (-1)^n \int_0^1 f(x)\left[\cos\bar{\lambda}_n\left(\frac{1}{2}-x\right) - \sin\bar{\lambda}_n\left(\frac{1}{2}-x\right)\right] dx =$$

$$= (-1)^n \cdot \int_0^1 f(x)\cos\bar{\lambda}_n\left(\frac{1}{2}-x\right)dx - (-1)^n \int_0^1 f(x)\sin\bar{\lambda}_n\left(\frac{1}{2}-x\right)dx;$$

$$\int_0^1 f(x)\cos\bar{\lambda}_n\left(\frac{1}{2}-x\right)dx = \begin{vmatrix} t = \frac{1}{2} - x \\ dt = -dx \end{vmatrix} = \int_{-\frac{1}{2}}^{\frac{1}{2}} f\left(\frac{1}{2}-t\right)\cos\bar{\lambda}_n t(-dt) =$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} f\left(\frac{1}{2} - t\right) [\cos 2n\pi t \ \cos 2\varphi t - \sin 2n\pi t \sin 2\varphi t] dt =$$

$$= \left| x = 2\pi t, \qquad t = \frac{2}{2\pi}, \qquad dt = \frac{dx}{2\pi} \right| =$$

$$= \int_{-\pi}^{+\pi} f\left(\frac{1}{2} - \frac{x}{2\pi}\right) \left[\cos nx \cdot \cos \frac{\varphi x}{\pi} - \sin nx \cdot \sin \frac{\varphi x}{\pi}\right] \frac{dx}{2\pi} =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} f\left(\frac{1}{2} - \frac{x}{2\pi}\right) \cos \frac{\varphi x}{\pi} \cos nx \, dx -$$

$$- \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{1}{2} - \frac{x}{2\pi}\right) \sin \frac{\varphi x}{\pi} \cdot \sin nx \, dx;$$

The system

$$\frac{1}{\sqrt{2\pi}}, \qquad \frac{1}{\sqrt{\pi}} \sin x, \dots, \frac{\cos nx}{\sqrt{\pi}}, \qquad \frac{\sin nx}{\sqrt{\pi}}, \dots$$

forms orthonormal basis of the space $H = L^2(0,1)$. Based on this fact, we estimate the coefficients $a_n$and b_n(see (22)).

$$a_n = \frac{1}{2\sqrt{\pi}} \int_{-\pi}^{+\pi} f\left(\frac{1}{2} - \frac{x}{2\pi}\right) \cos \frac{\varphi x}{\pi} \times \frac{\cos nx}{\sqrt{\pi}} -$$

$$- \frac{1}{2\sqrt{\pi}} \int_{-\pi}^{+\pi} f\left(\frac{1}{2} - \frac{x}{2\pi}\right) \sin \frac{\varphi x}{\pi} \times \frac{\sin nx}{\sqrt{\pi}} dx,$$

$$Rea_n = \frac{1}{2\sqrt{\pi}} \int_{-\pi}^{+\pi} Re\, f\left(\frac{1}{2} - \frac{x}{2\pi}\right) \cos \frac{\varphi x}{\pi} \times \frac{\cos nx}{\sqrt{\pi}} dx -$$

$$- \frac{1}{2\sqrt{\pi}} \int_{-\pi}^{+\pi} Ref\left(\frac{1}{2} - \frac{x}{2\pi}\right) \sin \frac{\varphi x}{\pi} \times \frac{\sin nx}{\sqrt{\pi}} dx, \qquad n \neq 0$$

Let

$$u(x) = \frac{Ref\left(\frac{1}{2} - \frac{x}{2\pi}\right) \cos \frac{\varphi x}{\pi}}{2\sqrt{\pi}}, v(x) = \frac{1}{2\sqrt{\pi}} Ref\left(\frac{1}{2} - \frac{x}{2\pi}\right) \sin \frac{\varphi x}{\pi}.$$

Then

$$Rea_n = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{+\pi} u(x) \cos nx \, dx - \frac{1}{\sqrt{\pi}} \int_{-\pi}^{+\pi} v(x) \sin nx \, dx = \alpha_n - \beta_n, n \neq 0$$

If $n = 0$, we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f\left(\frac{1}{2} - \frac{x}{2\pi}\right) \cos \frac{\varphi x}{\pi} dx,$$

$$Rea_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} Ref\left(\frac{1}{2} - \frac{x}{2\pi}\right) \cos \frac{\varphi x}{\pi} dx,$$

$$\alpha_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} u(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} \frac{Ref\left(\frac{1}{2} - \frac{x}{2\pi}\right) \cos \frac{\varphi x}{\pi}}{2\sqrt{\pi}} dx =$$

$$= \frac{1}{\sqrt{2}} \times \frac{1}{2\pi} \int_{-\pi}^{+\pi} Ref\left(\frac{1}{2} - \frac{x}{2\pi}\right) \cos \frac{\varphi x}{\pi} dx = \frac{Rea_0}{\sqrt{2}}, \Rightarrow Rea_0 = \alpha_0 \sqrt{2}.$$

Further,

$$|Rea_n| \le |\alpha_n| + |\beta_n|, \Rightarrow$$
$$|Rea_n|^2 \le (|\alpha_n| + |\beta_n|)^2 \le |\alpha_n|^2 + 2|\alpha_n| \times |\beta_n| + |\beta_n|^2 \le$$
$$\le |\alpha_n|^2 + |\beta_n|^2 + |\alpha_n|^2 + |\beta_n|^2 \le 2(|\alpha_n|^2 + |\beta_n|^2),$$
$$|Rea_0| \le \sqrt{2}\,|\alpha_0|, \qquad |Rea_0|^2 \le 2|\alpha_0|^2.$$

Consequently,

$$\sum_{n=0}^{\infty}|Rea_n|^2 \le 2\left(\sum_{n=0}^{\infty}|\alpha_n|^2 + \sum_{n=1}^{\infty}|\beta_n|^2\right) \le 2(\|u\|^2 + \|v\|^2) < \infty.$$

Similarly, we have

$$\sum_{n=0}^{\infty}|Ima_n|^2 < \infty.$$

Therefore,

$$\sum_{n=0}^{\infty}|a_n|^2 = \sum_{n=0}^{\infty}|Ima_n|^2 + |Rea_n|^2 < \infty.$$

Estimation of the series $\sum_{n=0}^{\infty}|b_n|^2$ is carried out similarly.

We have proved the main Theorem 3.1., and Theorem 3.2 is its corollary.

3. Research Results.

Theorem 3.1. Suppose that

$$\alpha^4 - \beta^4 \ne 0, \tag{8}$$

then the system of eigenfunctions of the boundary value problem

$$y'(1-x) = \lambda y(x); x \in (0,1), \tag{5}$$
$$\alpha y(0) + \beta y(1) = 0 \tag{6}$$

forms Riesz basis in the space $L^2(0,1)$, i.e. we have

$$f(x) = \sum_{-\infty}^{+\infty}(f, z_n)\, y_n(x),$$

Converging in the space $L^2(0,1)$, where

$$y_n(x) = (-1)^n\left[\cos\lambda_n\left(\frac{1}{2} - x\right) - \sin\lambda_n\left(\frac{1}{2} - x\right)\right],$$
$$z_n(x) = (-1)^n\left[\cos\overline{\lambda_n}\left(\frac{1}{2} - x\right) - \sin\overline{\lambda_n}\left(\frac{1}{2} - x\right)\right],$$
$$\lambda_n = 2n\pi + 2arctg\frac{\alpha+\beta}{\alpha-\beta},\ n = 0, \pm1, \pm2, \dots \tag{9}$$

and $f(x)$ is an arbitrary element in the space $L^2(0,1)$.

Theorem 3.2. If $\alpha^4 - \beta^4 \ne 0$, then for any element $u(x) \in D(A)$ we have the spectral expansion

$$Au = \sum_{-\infty}^{+\infty}\lambda_n(u, z_n)\, y_n(x) \tag{11}$$

converging in the space $L^2(0,1)$, where

$$\lambda_n = 2n\pi + 2arctg\frac{\alpha+\beta}{\alpha-\beta},\ n = 0, \pm1, \pm2, \dots \tag{9}$$
$$z_n(x) = (-1)^n\left[\cos\overline{\lambda_n}\left(\tfrac{1}{2} - x\right) - \sin\overline{\lambda_n}\left(\tfrac{1}{2} - x\right)\right], \tag{10$^+$}$$
$$y_n(x) = (-1)^n\left[\cos\lambda_n\left(\tfrac{1}{2} - x\right) - \sin\lambda_n\left(\tfrac{1}{2} - x\right)\right]. \tag{10}$$

Theorem 3.2. is the main result of this work.

4. Discussion.

Formula (11) is not possible if there are associated vectors, the well-known Kesselman - Mikhailov theorem [61- [62] states that, not the system of eigenvectors, the system of root vectors is basic, and this is significance of the results of this work. Formula (11) can find application in electrical engineering, information theory, crystallography, and signal transmission theory. It can be useful in study various boundary value problems by the method of variables separation.

5. Conclusion.

1) Operator (1) - (2) is not semi-bounded;

2) There is an alternative: either the boundary value problem (5) - (6) is Volterra i.e. has no eigenvalues, or the system of its eigenvectors forms a Riesz basis of the space $L^2(0,1)$.

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БІРІНШІ РЕТТІ ФУНКЦИОНАЛ-ДИФФЕРЕНЦИАЛ ОПЕРАТОРДЫҢ СПЕКТРӘЛДІ ТАРАЛЫМЫ

Аннотация. Бұл еңдекте аргументі ауытқыған бірінші ретті дифференциалдық теңдеудің спектрәлдік қасиеттері зерттелді.Шекаралық есептің спектрәлдік қасиеттері ретінде біз оның меншікті және олармен еншілес функцияларының системасының толымдылығы мен базистігін,сондайақ, оның вөлтерлігін таныймыз.

Түйін сөздер: аргументі ауытқыған теңдеу,толымдылық,базистік, вөлтерлік, Штурм -Лиувиллдің операторы, Рисстің базисі.

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СПЕКТРАЛЬНОЕ РАЗЛОЖЕНИЕ ФУНКЦИОНАЛЬНО-ДИФФЕРЕНЦИАЛЬНОГО ОПЕРАТОРА ПЕРВОГО ПОРЯДКА

Аннотация. В настоящей работе получено спектральное разложение функционально-дифференциального оператора первого порядка,с помощью прямого доказательства полноты системы системы собственных функций и теоремы Н.К.Бари.

Ключевые слова:уравнение с отклоняющимся аргументом, полнота, базисность, вольтерровость, операторы Штурма-Лиувилля, базис Рисса.

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REFERENCES

[1] Kato T. Perturbation Theory of Linear Operators M. Mir. 1972. 740p.

[2] Myshkis A.D. Linear Differential Equations with delayed argument, M. 1972. 352 p.

[3] Elsgolts L.E. and Norkin S.B. Introduction to theory of differential equations with deviating argument, M. 1971. 296 p.

[4] Norkin S.B. Differential Equations with Delayed Argument, M. 1965. 356 p.

[5] Kalmenov T.Sh., Akhmetova S. and Shaldanbaev A.Sh. To spectral theory of equations with deviating argument // Mathematical Journal, Almaty. 2004. V. 4, No. 3. P. 41-48.

[6] Ibraimkulov A.M. On spectral properties of boundary value problem for an equation with a deviating argument // News of AS.Kaz.SSR, ser.phys.-math. 1988. No. 3. Pp. 22-25.

[7] Akhmetova S.T. Spectral properties of boundary value problems for one class of differential equations with deviating argument: diss. ... cand. physical and mathematical sciences: 01.01.02. Shymkent, 2004. 96 p.

[8] Sadybekov M. A. and Sarsenbi A. M. Solutions of basic spectral problems of all boundary value problems for one first-order differential equation with a deviating argument. Uzbek Mathematical Journal. 2007. No. 3. Pp. 49-53.

[9] Sarsenbi A.M. On eigenvalues of degrees of linear operators. Vestnik L. N. Gumilyov ENU. 2006. No. 4 (50).- Pp. 20-24.

[10] Sadybekov M. A. and Sarsenbi A.M. On notion of regularity of boundary value problems for a second-order differential equation with a deviating argument// Mathematical Journal. 2007. v. 7, No. 1 (18). Pp. 82-88.

[11] Sarsenbi A.M. On asymptotic behavior of eigenvalues of boundary value problems for a second-order differential equation with a deviating argument// Izvestiya NAS RK, Ser. Physical and Math.2007. No. 1. Pp. 49-52.

[12] Sadybekov M. A. and Sarsenbi A. M. Unconditional convergence of spectral expansions associated with a second-order differential equation with a deviating argument// Vestnik KazNU. 2006. No. 2 (49). Pp. 48-54.

[13] Sarsenbi A.M. On Reisz basis of a system of eigenfunctions of one boundary value problem for the equation $-u''(-x) = \lambda u(x)$ // Izvestiya NAS RK. 2007. Ser. Phys.-Math. No. 3. Pp. 32-34.

[14] Sarsenbi A. M. Irregular boundary value conditions in the case of differential equations with deviating argument //Reports of NAS RK. 2007. No. 1. Pp. 12-16.

[15] W.Watkins. Modified Wiener equations. Int. J. Math. Math. Sci.Vol-27. p: 347-356. 2001y.

[16] W.T.Watkins. Asymptotic properties of differential equations with involutions. Int. J.Pure Appl. Math. Vol 44. p: 485-492, 2008 y.

[17] Burlutskaya M. Sh. and Khromov A. P. Foundation of the Fourier method in mixed problems with involution. // Izv. Sarat. Univ. New ser. Ser. Of Math. Mech. and Inf.2011. Vol. 11, No. 4. pp. 3-12.

[18] Burlutskaya M. Sh. and Khromov A. P. Fourier Method in a mixed problem for a first-order partial differential equation with involution.// J. Comput. Math. and Math. Phys. 2011. v. 51, No. 12. Pp. 2233-2246.

[19] Keldysh M. V. On completeness of eigenfunctions of certain classes of nonselfadjoint linear operators // Uspekhi Math. Nauk. - 1971. - Vol. 26, No. 4 (160). pp. 15-41.

[20] Il'in V. A. Existence of a reduced system of eigenand associated functions for a nonselfadjoint ordinary differential operator // Number theory, mathematical analysis and their applications, Trudy Mat. Inst. Steklov. 1976. 142. P. 148–155.

[21] Il'in V. A. Necessary and sufficient conditions of basis property and equivalent convergence with trigonometric series for spectral expansions, 1 / / Differential Equations. 1980. Vol. 16, No. 5. Pp. 771-794.

[22] Il'in V. A. Necessary and sufficient conditions of basis property and equivalent convergence with trigonometric series for spectral expansions, 2 / / Differential Equations. 1980. Vol. 16, No. 6. Pp. 980-1009.

[23] Il'in V. A. Necessary and sufficient conditions of basis property in L_2 and equivalent convergence with trigonometric series for spectral expansions and expansions by exponent systems// DAN SSSR. 1983. v. 273, No. 4. Pp. 784-793

[24] Il'in V. A. On unconditional basis property on a closed interval of systems of eigenfunctions and associated functions of the second order differential operator// DAS SSSR. 1983. Vol. 273, No. 5. Pp. 1048-1053.

[25] Il'in V. A. On relationship between the type of boundary value conditions and properties of basicity and equivalent convergence with a trigonometric series for expansions on root functions of a non-self-adjoint differential operator// Differential Equations. 1994. Vol. 30, No. 9. Pp. 1516-1529.

[26] Otelbaev M. Estimates of S-numbers and conditions of completeness of a root vector system of a non-self-adjoint Sturm-Liouville operator, Math. Notes, 1979, Vol. 25, No. 3, pp. 409-418.

[27] Shkalikov A. A. Boundary value problems for ordinary differential equations with parameter in boundary conditions// Trudy sem.by I. G. Petrovsky. 1983. Vol. 9. Pp. 190-229.

[28] Gomilko A.M. and Radzievsky G. V. Basis properties of eigenfunctions of a regular boundary value problem for a vector functional-differential equation// Differential Equations. 1991. Vol. 27, No. 3. Pp. 385-395.

[29] Il'in V.A., Kritskov L. V. Properties of spectral expansions corresponding to nonselfadjoint differential operators // J. Math. Sci. (NY). – 2003. 116, N 5. P. 3489–3550.

[30]Il'in V. A. and Kritskov L. V. Properties of spectral expansions corresponding to non-self-adjoint operators, Functional analysis. Results of science and technology. Ser.: Res.Math. and its App. Them. review., M.: VINITI, 2006, Vol. 96, pp. 5-105.

[31] Shaldanbaev A. Sh. and Akhmetova S. T. On completeness of eigenvectors of the Cauchy problem. Republican scientific journal "Science and Education SK" № 27, 2002. pp. 58-62.

[32]Akhmetova S. T., On Bitsadze - Samarsky problem for wave equation, Mathematical Journal, Almaty. 2003, Vol. 3. No. 2 (8), pp. 15-18.

[33] Shaldanbayev A. Sh. Criteria of Volterra property for differential operator of the first order with deviating argument, Vestnik of Karaganda University, series "Mathematics", № 3(47)/2007, pp. 39-43.

[34]Sadybekov M. A. and Sarsenbi A. M. Unconditional convergence of spectral expansions associated with a second-order differential equation with deviating argument. // Vestnik KAZNU. 2006. No. 2 (49). Pp. 48-54.

[35]Sadybekov M. A. and Sarsenbi A. M. Solution of main spectral problems of all boundary value problems for a first-order one-dimensional differential equation with deviating argument. // Uzbek Mathematical Journal - 2007.- No. 3.- pp. 88-94.

[36]Sadybekov M. A. and Sarsenbi A.M. On notion of regularity of boundary value problems for a second-order differential equation with deviating argument. // Mathematical Journal. 2007. V. 7, No. 1(18). Pp. 82-88.

[37]Sadybekov M. A. and Sarsenbi A. M. Criterion on basis property of system of eigenfunctions of multiple differentiation operator with involution.// Differential Equation. 2012. v. 48, No. 8. Pp. 1126-1132.

[38] Kopzhasarova A.A, Sarsenbi A.M. Basis Properties of Eigenfunctions of Second-Order Differential Operators with Involution//Abstract and Applied Analysis. 2012. Volume 2012, Article ID 576843. 6 pages doi:10.1155/2012/576843.

[39] Kopzhassarova A. A., Lukashov A. L., Sarsenbi A. M. Spectral properties of non-self-adjoint perturbations for a spectral problem with involution // Abstr. Appl. Anal. 2012. Pp. 1–5.
[40] Sarsenbi A.M.; Tengaeva A.A. On the basis properties of root functions of two generalized eigenvalue problems// Differential Equations. 2012. vol 48, no. 2. pp. 306–308.
[41] Imanbaeva A. B., Kopzhasarova A. A. and Shaldanbaev A. Sh. Asymptotic expansion of solution of singularly perturbed Cauchy problem for a system of ordinary differential equations with constant coefficients. "Izvestiya NAS RK. Series phys. and Math.", 2017, №5, 112 -127.
[42] Kopzhasarova A. A., Shaldanbaev A. sh. and Imanbaeva A. B. Solution of singularly perturbed Cauchy problem by similarity method. "Izvestiya NAS RK. Series Phys.-Math.", 2017, №5, 127-134.
[43] Amir Sh. Shaldanbayev, and Manat T. Shomanbayeva, Solution of singularly perturbed Cauchy problem for ordinary differential equation of second order with constant coefficients by Fourier method, Citation: AIP Conference Proceedings 1880, 040017 (2017); doi: 10.1063/1.5000633 View online: http://dx.doi.org/10.1063/1.5000633 View Table of Contents: http://aip.scitation.org/toc/apc/1880/1 Published by the American Institute of Physics.
[44] Akylbayev M. I., Beisebaeva A. and Shaldanbayev A. Sh. On periodic solution of the Gurss problem for a wave equation of a special form with variable coefficients (in English), Izvestiya NAS RK. Series Phys. and Math., No. 1. 2018, p.34-50
[45] A. A. Shaldanbaeva, M.I. Akylbayev, A. Sh. Shaldanbaev, A.Zh. Beisebaeva, The spectral decomposition of cauchy problem's solution for Laplace equation// N e w s of the National Academy of Sciences of the Republic of Kazakhstan, Physico-mathematical Series, ISSN 1991-346X Volume 5, Number 321 (2018), 75 – 87. https://doi.org/10.32014/2018.2518-1726.10
[46] A.Sh. Shaldanbayev, A.A. Shaldanbayeva, B.A. Shaldanbay, On projectional orthogonal basis of a linear non-self - adjoint operator, N e w s of the national academy of sciences of the republic of Kazakhstan, Physico-mathematical series, ISSN 1991-346X https://doi.org/10.32014/2019.2518-1726.15,Volume 2, Number 324 (2019), 79 – 89.
[47] M.I.Akylbayev, A.Sh.Shaldanbayev, I.Orazov, A.Beysebayeva, About single operator method of solution of a singularly perturbed Cauchy problem for an ordinary differential equation n – order,// N e w s of the national academy of sciences of the republic of Kazakhstan, Physico-mathematical series, ISSN 1991-346X Volume 2, Number 324 (2019), 17 – 36. https://doi.org/10.32014/2019.2518-1726.8
[48] Ashyralyev A., Sarsenbi A. M. Well-posedness of an elliptic equations with an involution // EJDE. 2015. 284. P. 1–8.
[49] Kritskov L. V., Sarsenbi A. M. Spectral properties of a nonlocal problem for the differential equation with involution // Differ. Equ. 2015. 51, N 8. P. 984– 990.
[50] Kurdyumov V. P. On Riesz bases of eigenfunction of 2-nd order differential operator with involution and integral boundary conditions // Izv. Saratov Univ. (N.S.), Ser. Math. Mech. Inform. 2015. 15, N 4. P. 392–405.
[51] Kirane M., Al-Salti N. Inverse problems for a nonlocal wave equation with an involution perturbation // J. Nonlinear Sci. Appl. 2016. 9. P. 1243–1251.
[52] Ya. O. Baranetskij, L. I. Kolyasa, Boundary value problem for second order differentialoperator equation with involution// Вісник національного університету, "Львівська політехніка","Фізико-математичні науки", № 871, 2017, pp. 20–26.
[53] Kritskov L.V., Sarsenbi A.M.,Equiconvergence property for spectral expansions related to perturbations of the operator $-u''(-x)$ with initial data // Filomat, 2018, vol. 32, № 3, pp. 1069-1078 .
[54] Kritskov L.V., Sadybekov M.A. and Sarsenbi A.M.,Nonlocal spectral problem for a second-order differential operator with an involution// BULLETIN OF THE KARAGANDA UNIVERSITY-MATHEMATICS, 2018 № 3 (91), pp. 53-60.
[55] Kritskov L.V., Sadybekov M.A., Sarsenbi A.M., Properties in Lp of root functions for a nonlocal problem with involution, Turkish Journal of Mathematics, Scientific and Technical research Council of Turkey - TUBITAK/Turkiye Bilimsel ve Teknik Arastirma Kurumu (Turkey), 2019, vol. 43, pp. 393-401.
[56] Vladykina V. E. and Shkalikov A. A., Spectral properties of ordinary differential operators with involution// Reports of the Academy of Sciences, 2019, vol. 484, No. 1, pp. 12-17.
[57] Shaldanbaev A. Sh. and Kulazhanov G. S. on spectral properties of one model operator with deviating argument// International conference "VEKUA-100", Novosibirsk 2007.
[58] Bari N. K. Biorthogonal systems and bases in Hilbert space, // Uch. Zap. MSU- 1951. Vol. 4, ser.148. pp. 69-106.
[59] Gokhberg I. S. and Krane M. G. Introduction to theory of linear non-self-adjoint operators in Hilbert space, M.: Nauka, 1965, 448 p.
[60] Naimark M. A. Linear Differential Operators. M.: Nauka, 1969. 528 p.
[61] Kesselman G. M. On unconditional convergence of expansions by eigenfunctions of some differential operators // Izv. Mathematics. - 1964.- No. 2 (39). - pp. 82-93.
[62] Mikhailov V. P. On Riezs basis in L2(0;1) // DAN USSR. 1962. Vol. 144, No. 5. Pp. 981-984.
[63] Lidsky V. B. Non-self-adjoint operators having trace. DAN USSR, 1959, vol. 125. No. 3, pp. 485-488.

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MULTIPERIODIC SOLUTIONS OF LINEAR SYSTEMS INTEGRO-DIFFERENTIAL EQUATIONS WITH D_c-OPERATOR AND ε-PERIOD OF HEREDITARY

Abstract. The article explores the questions of the initial problem and the problem of the multiperiodicity solutions of linear systems integro-differential equations with an operator of the form $D_c = \partial/\partial\tau + c_1\,\partial/\partial t_1 + \ldots + c_m\,\partial/\partial t_m$, $c = (c_1, \ldots, c_m) - const$ and with finite hereditary period $\varepsilon = const > 0$ that describe hereditary phenomena. Along with the equation of zeros of the operator D_c are considered linear systems of homogeneous and inhomogeneous integro-differential equations, sufficient conditions are established for the unique solvability of the initial problems for them, both necessary and sufficient conditions of multiperiodic existence are obtained by (τ, t) with periods (θ, ω) of the solutions. The integral representations of multiperiodic solutions of linear inhomogeneous systems are determined 1) in the particular case when the corresponding homogeneous systems have exponential dichotomy and 2) in the general case when the homogeneous systems do not have multiperiodic solutions, except for the trivial one.

Key words: integro-differential equation, hereditary, fluctuation, multiperiodic solution.

1. Problem statement.

In this paper, we've researched the problem of the existence of (θ, ω)-periodic solutions $u(\tau, t)$ by $(\tau, t) = (\tau, t_1, \ldots, t_m) \in R \times R \times \cdots \times R = R \times R^m$ systems of

$$D_c u(\tau,t) = A(\tau,t)u(\tau,t) + \int_{\tau-\varepsilon}^{\tau} K(\tau,t,s,t - c\tau + cs)u(s,t - c\tau + cs)ds + f(\tau,t) \quad (1.1)$$

with a differentiation operator D_c of the form

$$D_c = \partial/\partial\tau + \langle c, \partial/\partial t\rangle, \quad (1.2)$$

that turns into the operator of the total derivative $d/d\tau$ along the characteristics $t = c\tau - cs + \sigma$ with initial data $(s, \sigma) \in R \times R^m$, where $R = (-\infty, +\infty)$, $c = (c_1, \ldots, c_m)$ is constant vector with non-zero coordinates c_j, $j = \overline{1, m}$, $\partial/\partial t = (\partial/\partial t_1, \ldots, \partial/\partial t_m)$ is vector, $\langle c, \partial/\partial t\rangle$ is the scalar product of vectors, $A(\tau, t)$ and $K(\tau, t, s, \sigma)$ are given $n \times n$-matrices, $f(\tau, t)$ is n-vector-function, $(\theta, \omega) = (\theta, \omega_1, \ldots, \omega_m)$ is vector-period with rationally incommensurable coordinates, ε is positive constant.

The problem of this kind involves the research problems of hereditary vibrations in mechanics and electromagnetism. For example, if the oscillation phenomenon is hereditary in nature, then the equation of motion of the string at a known moment $m(\tau)$ is set by changing the angle of string torsion $\omega(\tau)$, subordinated to the ratio

$$m(\tau)-\mu\frac{d^2\omega(\tau)}{d\tau^2}=h\omega(\tau)+\int_{\tau-\varepsilon}^{\tau}\varphi(\tau,s)\omega(s)ds, \qquad (1.3)$$

where μ and h are constants and ε is the hereditary period of the vibrational phenomenon.

It is also known that the hereditary biological phenomenon "predator-prey" - (N_1, N_2) is related by the law of oscillations described by the system of equations

$$\begin{aligned}\frac{dN_1(\tau)}{d\tau}&=N_1(\tau)\left\{\varepsilon_1-\gamma_1N_2(\tau)-\int_{\tau-\varepsilon}^{\tau}F_1(\tau-s)N_2(s)ds\right\},\\ \frac{dN_2(\tau)}{d\tau}&=N_2(\tau)\left\{-\varepsilon_2+\gamma_2N_1(\tau)+\int_{\tau-\varepsilon}^{\tau}F_2(\tau-s)N_1(s)ds\right\}.\end{aligned} \qquad (1.4)$$

Where $\varepsilon_1,\varepsilon_2$ and γ_1,γ_2 are constants, F_1 and F_2 are functions vanishing zero at $\tau-s\geq\varepsilon$, ε is the period of hereditary nature of the biological phenomenon under consideration.

Obviously, the above integro-differential equations (1.3) and (1.4) are particular cases of the mathematical model hereditary phenomena described by the system of equations

$$\frac{dx}{dt}=P(\tau)x(\tau)+\int_{\tau-\varepsilon}^{\tau}Q(\tau,s)x(s)ds+\psi(\tau) \qquad (1.5)$$

relatively sought n-vector-function with given $n\times n$-matrices $P(\tau)$ and $Q(\tau,s)$ and with n-vector-function $\psi(\tau)$, where $\varepsilon>0$ is a constant. Since the process is oscillatory, as a rule, the matrix $P(\tau)$ and the vector function $\psi(\tau)$ are almost periodic in general case and the kernel $Q(\tau,s)$ has the property of diagonal periodicity by $(\tau,s)\in R\times R$.

In particular, the indicated input data of system (1.5) are quasiperiodic by $\tau\in R$ with a frequency basis $\nu_0=\theta^{-1},\nu_1=\omega_1^{-1},\ldots,\nu_m=\omega_m^{-1}$, then in the theory of fluctuations, the question of the existence of quasiperiodic solutions $x(\tau)$ of system (1.5) with a modified frequency basis is important $\widetilde{\nu}_0=\theta^{-1},\widetilde{\nu}_1=c_1\omega_1^{-1},\ldots,\widetilde{\nu}_m=c_m\omega_m^{-1}$ and we set $\varepsilon<\theta=\omega_0<\omega_1<\ldots<\omega_m$.

An important role in solving this problem is played by the well-known theorem of G. Bohr on the deep connection between quasiperiodic functions and periodic functions of many variables (multiperiodic functions). According to this theorem, matrix-vector functions are defined $A=A(\tau,t), K=K(\tau,t,s,\sigma)$, $\sigma=t-c\tau+cs$, $f=f(\tau,t)$, $u=u(\tau,t)$ with properties of $A|_{t=c\tau}=P(\tau)$, $K|_{t=c\tau}=Q(\tau,s)$, $f|_{t=c\tau}=\psi(\tau)$, $u|_{t=c\tau}=x(\tau)$ and the operator $d/d\tau$ is replaced by a differentiation operator D_c of the form (1.2).

Thus, the problem of quasiperiodic fluctuations in systems (1.5) becomes equivalent to the problem on the existence of (θ,ω)-periodic by (τ,t) solutions $u(\tau,t)$ of the system partial integro-differential equations of the form (1.1) with differentiation operator (1.2).

The above examples of problems on string vibrations and fluctuations in the numbers of two species living together associated with the task indicate the relevance of the latter, in terms of its applicability in life. Along with this, it is worth paying special attention to the fact that the methods of researching multiperiodic solutions of integro-differential equations and systems of such partial differential equations belong to a poorly studied section of mathematics. Therefore, the development of methods of the theory of multiperiodic solutions of partial differential integro-differential equations is of special scientific interest.

In the present work are investigated to obtain conditions for the existence of multiperiodic solutions of linear systems integro-differential equations with a given differentiation operator D_c. To achieve this goal, the initial problems for the considered systems of equations are solved from the beginning, and then the necessary and sufficient conditions for the existence of multiperiodic solutions of linear systems equations are established. The integral structures of solutions linear inhomogeneous systems with the property of uniqueness are determined.

The theoretical basis of this research is based on the work of several authors. As noted above, taking into account the hereditary nature of various processes of physics, mechanics, and biology leads to the consideration of integro-differential equations [1–16], especially to the research of problems for them related to the theory of periodic fluctuations [8, 9, 12, 13]. If the heredity of the phenomenon is limited to a finite period ε of time τ, then the hereditary effect is specified by the integral operator with variable limits from $\tau-\varepsilon$ to τ.

Integro-differential equations describing phenomena with such hereditary effects are considered in [5, 6, 12, 14]. The various processes of hereditary continuum mechanics are described by partial integro-differential equations, the study of which began with the works [1, 2, 4].

The work of many authors is devoted to finding effective signs of solvability and the construction of constructive methods for researching problems for systems of differential equations, we note only [17, 18].

The research of multi-frequency oscillations led to the concept of multidimensional time. In this connection, of the theory solutions of partial differential equations that are periodic in multidimensional time is being developed, both in time and in space independent variables [19–35]. It is known that the system of canonical Hamilton equations, under fairly general conditions, can be solved by the Jacobi method, the essence of which is the transition from its integration to the integration of a partial differential equation. A similar approach is implemented in [19], where quasiperiodic solutions of ordinary differential equations are studied with a transition to the research of multiperiodic solutions of partial differential equations. This method was developed in [20–30] with its extension to the solution of a number of oscillation problems in systems of integro-differential equations.

In this research, it is examined for the first time that the problem of the existence multiperiodic solutions of systems integro-differential equations with a special differentiation operator D_c, describing hereditary processes with a finite period ε of hereditary time τ.

In solving this problem, we encountered the problems associated with the multidimensionality of time; not developed general theory of such systems; determination of structures and integral representations of solutions of linear systems equations; extending the results of the linear case to the nonlinear case; the smoothness of the solutions integral equations equivalent to the problems under consideration, etc. These barriers to solving problems have been overcome due to the spread and development of the methods of works [31-35] used to solve similar problems for systems of differential equations.

2. Zeros of the differentiation operator and their multiperiodicity.

By the zero of the operator D_c we mean a smooth function $u=u(\tau,t)$ satisfying the equation of

$$D_c u = 0. \tag{2.1}$$

The linear function

$$t = h(\tau,\tau^0,t^0) \equiv t^0 + c\tau - c\tau^0 \tag{2.2}$$

is a general solution of the characteristic equation $dt/d\tau = c$ with the initial data (τ^0, t^0), and its integral obtained from equation (2.2) by relative solution t^0 type of the form

$$h(\tau^0,\tau,t) = t - c\tau + c\tau^0 \tag{2.3}$$

is the zero of the operator D_c satisfying condition $h(\tau^0,\tau,t)\big|_{\tau=\tau^0} = t$.

It is also easy to verify that if $\psi(t)$ is an any smoothness function $e=(1,\ldots,1)$, by $t=(t_1,\ldots,t_m)\in R\times\ldots\times R=R^m$, then the function

$$u(\tau^0,\tau,t)=\psi\left(h(\tau^0,\tau,t)\right) \tag{2.4}$$

is the zero of the operator D_c satisfying condition of $u\big|_{\tau=\tau^0}=\psi(t)$.

Since the function $\psi(t)$ is arbitrary in the class $C_t^{(e)}(R^m)$ of functions smoothness e by $t\in R^m$, relation (2.4) is a general formula of the zeros of the operator D_c.

In connection with the research of question on multiperiodicity of the zeros operator D_c, attention should be paid to the following properties of the characteristics $h(s,\tau,t)$ of operator D_c:

$$h(s+\theta,\tau+\theta,t)=h(s,\tau,t), \tag{2.5}$$

$$h(s,\tau+\theta,t)=h(s,\tau,t)-c\theta, \tag{2.6}$$

$$h(s,\tau,t+q\omega)=h(s,\tau,t)+q\omega, \tag{2.7}$$

which follow from the linearity of the function (2.3), where $q\omega=(q_1\omega_1,\ldots,q_m\omega_m)$, $q=(q_1,\ldots,q_m)\in Z\times\ldots\times Z=Z^m$, Z are set of integers.

If $u(\tau,t)$ is the zero of operator D_c (θ,ω)-periodic by (τ,t), then the initial function $u\big|_{\tau=\tau^0}=u^0(t)$ is ω-periodic by t:

$$u^0(t+q\omega)=u^0(t)\in C_t^{(e)}(R^m), q\in Z^m. \tag{2.8}$$

Therefore, condition (2.8) is a necessary condition for the (θ,ω)-periodicity of zero $u(\tau,t)\in C_{\tau,t}^{(1,e)}(R\times R^m)$.

Suppose that for zero $u(\tau,t)$ of the operator D_c the necessary condition is satisfied (2.8) for its (θ,ω)-periodicity by (τ,t). Then $u(\tau,t)$ according to formula (2.4) has the form of

$$u(\tau,t)=u^0\left(h(\tau^0,\tau,t)\right). \tag{2.9}$$

Obviously, by virtue of conditions (2.7) and (2.8), the researching zero (2.9) is ω-periodic by t. For a zero $u(\tau,t)$ to be θ-periodic by τ, we require that condition

$$u^0\left(h(\tau^0,\tau+\theta,t)\right)=u^0\left(h(\tau^0,\tau,t)-c\theta\right) \tag{2.10}$$

which holds by virtue of property (2.6).

From this it is clear that, under condition (2.8), relation (2.10) holds if only some integer vector $q^0\in Z^m$ is found and equality

$$c\theta+q^0\omega=0, \tag{2.11}$$

which means the commensurability of the $c\theta=(c_1\theta,\ldots,c_m\theta)$ and $\omega=(\omega_1,\ldots,\omega_m)$ vectors.

It should be noted here that condition (2.11) is required if the initial function $u^0(t)$ necessarily depends on the variable t. Otherwise, when $u^0=const$, condition (2.10) is performed automatically, conditions (2.11) are not needed.

Thus, if conditions (2.11) are not satisfied, then the (θ,ω)-periodic zero of the operator D_c are constant.

Obviously, due to the condition of (2.5), the zeros $u(\tau^0,\tau,t)$ of operator D_c form (2.4) have the property of diagonal θ-periodicity by (τ^0,τ): $u(\tau^0+\theta,\tau+\theta,t)=u(\tau^0,\tau,t)$.

The proof of this property follows from (2.5) and (2.4) based on direct verification.

The obtained results are summarized in the form of the following theorem.

Theorem 2.1. *1) If condition (2.11) is not satisfied, then only constants are the (θ,ω)-periodic zeros of the operator D_c and it does not have multiperiodic variables zeros. 2) If condition (2.11) is satisfied, then any zero of the operator D_c with an initial function of the form (2.8) is (θ,ω)-periodic, in particular, it can be any constant. 3) Any zero of the form (2.4) has the property of diagonal θ-periodicity by (τ^0,τ), and from its θ-periodicity zeros by τ follows its θ-periodicity by τ^0.*

Further, in conclusion, we note one more important group property of characteristic

$$h(\tau^0,\xi,h(\xi,\tau,t))=h(\tau^0,\tau,t), \tag{2.12}$$

necessary in the future, in justice, which can be verified by direct verification.

3. Linear homogeneous equations and their multiperiodic solutions.

We consider the initial problem for a linear homogeneous system

$$D_c u(\tau,t)=A(\tau,t)u(\tau,t)+\int_{\tau-\varepsilon}^{\tau}K(\tau,t,s,h(s,\tau,t))u(s,h(s,\tau,t))ds \tag{3.1}$$

with respect to the desired n-vector-function $u(\tau,t)$ with condition

$$u(\tau,t)\big|_{\tau=\tau^0}=u^0(t)\in C_t^{(e)}(R^m) \tag{3.1^0}$$

under assumptions of

$$A(\tau+\theta,t+q\omega)=A(\tau,t)\in C_{(\tau,t)}^{(0,2e)}(R\times R^m),\ q\in Z^m, \tag{3.2}$$

$$K(\tau+\theta,t+q\omega,s,\sigma)=K(\tau,t,s+\theta,\sigma+q\omega)=K(\tau,t,s,\sigma)\in$$
$$\in C_{\tau,t,s,\sigma}^{(0,2e,0,2e)}(R\times R^m\times R\times R^m),\ q\in Z^m \tag{3.3}$$

where $\tau^0\in R$.

It is obvious [19, 20, 21, 28, 29] that under condition of (3.2), using the method of successive approximations, we can construct a matricant $W(\tau^0,\tau,t)$ of the linear system of partial differential equations of the form

$$D_c w(\tau,t)=A(\tau,t)w(\tau,t), \tag{3.4}$$

which has property

$$D_c W(\tau^0,\tau,t)=A(\tau,t)W(\tau^0,\tau,t),\ W(\tau^0,\tau^0,t)=E, \tag{3.5}$$

$$D_c W^{-1}(\tau^0,\tau,t)=-W^{-1}(\tau^0,\tau,t)A(\tau,t), \tag{3.6}$$

$$W(\tau^0+\theta,\tau+\theta,t+q\omega)=W(\tau^0,\tau,t),\ q\in Z^m, \tag{3.7}$$

where E is the identity n-matrix.

Then, using the replacement of

$$u(\tau,t)=W(\tau^0,\tau,t)v(\tau,t) \tag{3.8}$$

system (3.1) is reduced to the form of integro-differential equation

$$D_c v(\tau,t)=\int_{\tau-\varepsilon}^{\tau} Q(\tau^0,\tau,t,s,h(s,\tau,t))v(s,h(s,\tau,t))ds \tag{3.9}$$

with the kernel

$$Q(\tau^0,\tau,t,s,\sigma)=W^{-1}(\tau^0,\tau,t)K(\tau,t,s,\sigma)W(\tau^0,s,\sigma), \tag{3.10}$$

which, due to the properties (2.5)-(2.7) of the characteristics $h(s,\tau,t)$, (3.3) of the kernel $K(\tau,t,s,\sigma)$ and (3.5)-(3.7) matricant $W(\tau^0,\tau,t)$, has the properties of multiperiodicity and smoothness of the form

$$\begin{gathered}Q(\tau^0+\theta,\tau+\theta,t+q\omega,s+\theta,h(s+\theta,\tau+\theta,t+q\omega))=\\ =Q(\tau^0,\tau,t,s,h(s,\tau,t))=Q(\tau^0,\tau,t,s,\sigma)\in\\ \in C^{(1,1,e,1,e)}_{\tau^0,\tau,t,s,\sigma}(R\times R\times R^m\times R\times R^m), q\in Z^m.\end{gathered} \tag{3.11}$$

Further, under condition (3.3), integrating along the characteristics: $\tau=\eta$, $t=h(\eta,\tau,t)$, using property (2.12) of the form $h(\xi,\eta,h(\eta,\tau,t))=h(\xi,\tau,t)$, from equation (3.9), using the method of successive approximations, we find its matrix solution $V(s,\tau,t)$ based on the integral equation

$$V(s,\tau,t)=E+\int_s^{\tau} d\eta \int_{\eta-\varepsilon}^{\eta} Q(s,\eta,h(\eta,\tau,t),\xi,h(\xi,\tau,t))V(s,\xi,h(\xi,\tau,t))d\xi, \tag{3.12}$$

and by virtue of properties (3.11) of the kernel Q of this equation, we easily have the following relation

$$V(s+\theta,\tau+\theta,t+q\omega)=V(s,\tau,t)\in C^{(1,1,e)}_{s,\tau,t}(R\times R\times R^m), q\in Z^m. \tag{3.13}$$

Obviously, by virtue of (3.12) and (3.13), we have

$$D_c V(s,\tau,t)=\int_{\tau-\varepsilon}^{\tau} Q(s,\tau,t,\xi,h(\xi,\tau,t))V(\xi,h(\xi,\tau,t))d\xi, \tag{3.14}$$

$$V(s,s,t)=E. \tag{3.14^0}$$

We note that the matrix A, the kernel K, and the period ε are such that the matrix $V(s,\tau,t)$ is invertible, moreover

$$D_c V^{-1}(s,\tau,t)=-V^{-1}(s,\tau,t)\cdot D_c V(s,\tau,t)\cdot V^{-1}(s,\tau,t). \tag{3.14$'$}$$

Then the matrix

$$U(s,\tau,t)=W(s,\tau,t)V(s,\tau,t), \tag{3.15}$$

constructed on the basis of the replacement (3.8) is invertible, satisfies the equation (3.1), becomes the identity matrix E at $\tau=s$, and has the property of diagonal θ-periodicity by (s,τ) and ω-periodicity by t:

$$D_c U(s,\tau,t)=A(\tau,t)U(s,\tau,t)+\int_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))U(\xi,h(\xi,\tau,t))d\xi,$$

$$U(s,s,t)=E, \tag{3.16}$$

$$U(s+\theta,\tau+\theta,t+q\omega)=U(s,\tau,t)\in C^{(1,1,e)}_{s,\tau,t}(R\times R\times R^m), q\in Z^m. \tag{3.17}$$

These properties of (3.16) and (3.17) matrix $U(s,\tau,t)$ are consequences of the properties (3.5)-(3.7) and (3.13), (3.14), (3.14^0) the matrices $W(s,\tau,t)$ and $V(s,\tau,t)$.

The matrix $U(s,\tau,t)$ can be called the resolving operator of the integro-differential system (3.1).

Theorem 3.1. *Let conditions (3.2) and (3.3) are satisfied. Then the solution $u(\tau^0,\tau,t)$ of the initial problem (3.1)-(3.1^0) is uniquely determined by the relation*

$$u(\tau^0,\tau,t)=U(\tau^0,\tau,t)u^0(h(\tau^0,\tau,t)). \tag{3.18}$$

Proof. By condition (3.1^0), in accordance with formulas (2.8) and (2.9), the vector function $u^0(h(\tau^0,\tau,t))$ is the zero of the operator D_c: $D_c u^0(h(\tau^0,\tau,t))\equiv 0$.

Given this, by virtue of relations (2.12), (3.16) and expression (3.18), we have

$$D_c u^0(h(\tau^0,\tau,t))=A(\tau,t)U(\tau^0,\tau,t)u^0(h(\tau^0,\tau,t))+$$

$$+\int_{\tau-\varepsilon}^{\tau}K(\tau,t,\xi,h(\xi,\tau,t))U(\xi,h(\xi,\tau,t))d\xi\cdot u^0(h(\tau^0,\tau,t))=$$

$$=A(\tau,t)u(\tau^0,\tau,t)+\int_{\tau-\varepsilon}^{\tau}K(\tau,t,\xi,h(\xi,\tau,t))U(\xi,h(\xi,\tau,t))u^0(h(\tau^0,\xi,h(\xi,\tau,t)))d\xi=$$

$$=A(\tau,t)u(\tau^0,\tau,t)+\int_{\tau-\varepsilon}^{\tau}K(\tau,t,\xi,h(\xi,\tau,t))u(\tau^0,\xi,h(\xi,\tau,t))d\xi.$$

Thus, we were convinced that the vector function (3.18) satisfies the system (3.1). By virtue of (2.3) and (3.16), at $\tau=\tau^0$ we have condition (3.1^0). The uniqueness of the solution (3.18) follows from the uniqueness of the definition of matrices $W(\tau^0,\tau,t)$ and $V(\tau^0,\tau,t)$.

The theorem is completely proved.

Now, after establishing the structure of the general solution (3.18) of system (3.1), we have the opportunity to research the multiperiodicity of its solutions.

Theorem 3.2. *Let the conditions of theorem 3.1 are satisfied. In order to the solution $u(\tau,t)$ of system (3.1) is (θ,ω)-periodic, it is necessary, that its initial function $u(0,t)=u^0(t)$ at $\tau=0$ should be ω-periodic continuously differentiable function of the variable $t\in R^m$:*

$$u^0(t+q\omega)=u^0(t)\in C_t^{e/\omega}(R^m),\ q\in Z^m. \tag{3.19}$$

Proof. Indeed, for $\tau^0=0$, from the formula for the general solution (3.18) of system (3.1) we have

$$u(\tau,t)=U(0,\tau,t)u^0(h(0,\tau,t)), \tag{3.20}$$

and it is (θ,ω)-periodic by (τ,t). Therefore, the condition is satisfied

$$u(\tau+\theta,t+q\omega)=u(\tau,t),\ q\in Z^m. \tag{3.21}$$

Then, in particular, from the set (3.21) we obtain relation of

$$u(\tau,t+q\omega)=u(\tau,t),\ q\in Z^m. \tag{3.22}$$

Substituting the representation (3.20) into the identity (3.22) we have

$$U(0,\tau,t+q\omega)u^0(h(0,\tau,t+q\omega))=U(0,\tau,t)u^0(h(0,\tau,t)).$$

Then, by virtue of properties (2.7) and (3.17), we obtain

$$U(0,\tau,t)u^0(h(0,\tau,t)+q\omega)=U(0,\tau,t)u^0(h(0,\tau,t)),\ q\in Z^m.$$

Further, setting the $t=0$ and taking into account (3.16), we have

$$u^0(t+q\omega)=u^0(t),\ q\in Z^m.$$

Thus, the identity (3.19) is proved. The smoothness of the initial functions $u^0(t)$ follows from the smoothness of the solution $u(\tau,t)$ of system (3.1) itself. This is what was required to prove.

Theorem 3.3. *In order for the solution $u(\tau,t)$ of system (3.1) for being ω-periodic by $t\in R^m$ under the conditions of theorem 3.2, it is necessary and sufficient for condition (3.19) be satisfied with respect to the initial function $u^0(t)$ for $\tau=0$.*

Proof. ◄ Necessity follows from theorem 3.2. To prove sufficiency, we show that relation (3.22) follows from condition (3.19). To do this, it suffices to use representation (3.20) and properties (2.7) and (3.17) of the characteristic and matricant, respectively. ►

Theorem 3.4. *In order for the solution $u(\tau,t)$ to be θ-periodic by $\tau\in R$ under the conditions of theorem 3.3, it is necessary and sufficient that the initial function $u^0(t)$ at $\tau=0$ be a ω-periodic solution of the linear ω-periodic by t functional difference system*

$$U(0,\theta,t)u^0(t-c\theta)=u^0(t) \tag{3.23}$$

with difference $\rho=c\theta$ by t.

◄ The condition of θ-periodicity by τ of the solution $u(\tau,t)$ has the form

$$u(\tau+\theta,t)=u(\tau,t),\ (\tau,t)\in R\times R^m\ . \tag{3.24}$$

By virtue of the uniqueness properties, the solutions of system (3.1) to satisfy condition (3.24) are necessary and sufficient for condition

$$u(\theta,t)=u(0,t) \tag{3.25}$$

to hold.

Then, using representation (3.20), we rewrite identity (3.25) in the form

$$U(0,\theta,t)u^0(h(0,\theta,t))=U(0,0,t)u^0(h(0,0,t)).$$

Hence, by virtue of properties (2.6) and (3.16), we have the necessary and sufficient condition (3.23).►

If $u_0(\tau,t)=u^0(h(0,\tau,t))$ is the (θ,ω)-periodic zero of the operator D_c, then the solution $u(\tau,t)$ of the form (3.20):

$$u(\tau,t)=U(0,\tau,t)u_0(\tau,t) \tag{3.26}$$

we call the solution ω-periodic by t generated by the (θ,ω)-periodic zero $u_0(\tau,t)$ of the differentiation operator D_c.

Theorem 3.5. *In order for the solution $u(\tau,t)$ to be (θ,ω)-periodic solution of system (3.1) generated by the (θ,ω)-periodic zero $u_0(\tau,t)$ of the operator D_c under the conditions of theorem 3.4, it is necessary and sufficient that the vector function $u_0(\tau,t)=v(t)$ be an eigenvector of the monodromy matrix $U(0,\theta,t)=V(t)$:*

$$[V(t)-E]v(t)=0. \tag{3.27}$$

◄ According to theorem 2.1, we have $u^0(t-c\theta)=u^0(t)$. Therefore, the necessary and sufficient condition (3.23) given by theorem 3.4 has the form

$$[U(0,\theta,t)-E]u^0(t)=0. \tag{3.28}$$

Obviously, $u^0(t)=u_0(0,t)=v(t)$. Then from relation (3.28) we have the condition of (θ,ω)-periodicity of solutions $u(\tau,t)$ from the class under consideration generated by the multiperiodic zeros of operator D_c.►

Note that if the commensurability condition (2.11) is not satisfied, then $v(t)$ becomes constant: $v=c^0-const$ and the condition (3.27) of multiperiodicity has the form

$$[V(t)-E]c^0=0,\ t\in R^m. \tag{3.29}$$

In order to avoid nonzero (θ,ω)-periodic solutions of system (3.1), in this case, it is sufficient to require that condition

$$\det[V(t)-E]\neq 0,\ t\in R^m. \tag{3.30}$$

be satisfied.

The research of multiperiodic solutions of the form (3.26) of system (3.1) is a separate interesting direction in the theory of multiperiodic solutions of such systems, which is based on conditions (3.27) - (3.30).

In many cases, there is necessary to clarify the conditions for the absence of multiperiodic solutions of systems of the form (3.1) other than the trivial $u=0$.

To do this, according to theorem 3.4, it is necessary to require that the ω-periodic functional-difference system (3.23) does not have a solution nonzero that is ω-periodic by t. In this regard, we assume that the resolving operator $U(\tau^0,\tau,t)$ of the system of integro-differential equations (3.1) satisfies condition

$$|U(s,\tau,t)|\leq a\,e^{-\alpha(\tau-s)},\ \tau\geq s \tag{3.31}$$

with the constants $a\geq 1$ and $\alpha>0$.

Then the matrix $U(0,\tau,t)$ at $\tau=0$ turns into the identity matrix E and at $\tau>0$, according to condition (3.31), decreases exponentially. Therefore, the monodromy matrix $U(0,\theta,t)=V(t)$ at all $t\in R^m$ satisfies condition

$$|V(t)|\leq b=const<1,\ \ t\in R^m, \tag{3.32}$$

where $b=a\,e^{-\alpha\theta}=const<1$.

Therefore, representing system (3.23) in the form

$$u^0(t)=V(t)u^0(t-c\theta) \tag{3.33}$$

and integrating it j times, we have

$$u^0(t)=V(t)V(t-c\theta)\ldots V(t-jc\theta)u^0(t-c\theta-jc\theta).$$

Estimating the last relation, on the basis of (3.32) we have inequality

$$|u^0(t)|\leq b^{j+1}|u^0(t-c\theta-jc\theta)|,\ \ 0<b<1.$$

Hence, passing to the limit at $j \to +\infty$, taking into account that the quantity $\left|u^0(t)\right|$ is bounded due to its ω-periodicity, we have $\left|u^0(t)\right| = 0$, that is, system (3.33) has only a zero ω-periodic solution.

Thus, the following theorem is proved.

Theorem 3.6. *In order for the system of integro-differential equations (3.1) has no multiperiodic solutions, except for the zero one under the conditions of theorem 3.4, the fulfillment of condition (3.31) is sufficient.*

Note that the proved theorem 3.6 remains valid if condition (3.31) is replaced by condition

$$\left|U(s,\tau,t)\right| \le a\, e^{\alpha(\tau-s)},\ \ \tau \le s \tag{3.34}$$

with constants $a \ge 1$ and $\alpha > 0$.

The more general than (3.31) and (3.34) the absence of condition a multiperiodic solution other than zero is the decomposability condition for a resolving operator $U(s,\tau,t)$ into the sum of two matrix solutions $U_-(s,\tau,t)$ and $U_+(s,\tau,t)$ of system (3.1) in the form

$$U(s,\tau,t) = U_-(s,\tau,t) + U_+(s,\tau,t), \tag{3.35}$$

$$D_c U_\mp(s,\tau,t) = A(\tau,t)U_\mp(s,\tau,t) + \int\limits_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))U_\mp(s,\xi,h(\xi,\tau,t))d\xi, \tag{3.36}$$

satisfying conditions

$$\left|U_-(s,\tau,t)\right| \le a\, e^{-\alpha(\tau-s)},\ \tau \ge s, \tag{3.37}$$

$$\left|U_+(s,\tau,t)\right| \le a\, e^{\alpha(\tau-s)},\ \tau \le s \tag{3.38}$$

with some constants $a \ge 1$ and $\alpha > 0$.

In particular, when one of the matrices U_- and U_+ is equal to zero, then we obtain either condition (3.31) or condition (3.34), respectively.

If conditions (3.35) - (3.38) are satisfied, they say that the resolving operator $U(s,\tau,t)$ has the property of exponential dichotomy.

We note that for system (3.1), the case of exponential dichotomy is possible when for the monodromy matrix U there exist projectors P_- and P_+ with the properties $P_- + P_+ = I$ is the identity operator, $P_+P_- = P_-P_+ = 0$ is the zero operator and $P_\mp(Uu) = P_\mp U \cdot P_\mp u$, where P_- projects the space of solutions onto the subspace of exponentially decreasing in norm of solutions, and P_+ - on the subspace of exponentially increasing solutions.

Then system

$$V(t)v(t - c\theta) = v(t) \tag{3.39}$$

is equivalent to system

$$V_-(t)v_-(t - c\theta) = v_-(t), \tag{3.40$_-$}$$

$$V(t)_+ v_+(t - c\theta) = v_+(t). \tag{3.40$_+$}$$

As above, it was justified that systems (3.40_-) and (3.40_+) have only zero multiperiodic solutions; therefore, system (3.39) also has only a zero solution, provided that conditions (3.35) - (3.38) are satisfied.

Thus, we can state a theorem that generalizes theorem 3.6 proved above.

Theorem 3.7. *Let conditions (3.2), (3.3), and (3.35) - (3.38) be satisfied. Then system (3.1) has no multiperiodic solutions, except for the trivial one.*

4. Linear inhomogeneous equations and their multiperiodic solutions.

We consider the linear inhomogeneous system of integro-differential equations

$$D_c u(\tau,t) = A(\tau,t)u(\tau,t) + \int_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))u(\xi,h(\xi,\tau,t))d\xi + f(\tau,t), \quad (4.1)$$

the corresponding system (3.1), where the $f(\tau,t)$ is given n-vector-function possessing property

$$f(\tau+\theta,t+q\omega) = f(\tau,t) \in C_{\tau,t}^{(0,e)}(R\times R^m),\ q\in Z^m. \quad (4.2)$$

Under the condition (4.2), we are posing the definition of a solution $u = u(\tau,t)$ of system (4.1) satisfying the initial condition

$$u\big|_{\tau=\tau^0} = u^0(t) \in C_t^{(e)}(R^m). \quad (4.1^0)$$

We begin the solution of this question about finding the particular solution $u^*(\tau^0,\tau,t)$ of system (4.1) with zero initial condition

$$u^*(\tau^0,\tau,t)\big|_{\tau=\tau_0} = 0. \quad (4.1^*)$$

We will seek this solution in the form

$$u^*(\tau^0,\tau,t) = \int_{\tau^0}^{\tau} U(s,\tau,t)v(s,h(s,\tau,t))ds \quad (4.3)$$

with an unknown, continuous, smooth by t at $(\tau,t)\in R\times R^m$ n-vector-function

$$v(\tau,t)\in C_{\tau,t}^{(0,e)}(R\times R^m), \quad (4.4)$$

where $U(s,\tau,t)$ is resolving operator of the homogeneous system (3.1).

Acting by the operator D_c on the vector function (4.3), taking into account (4.4), we have

$$D_c u^*(\tau^0,\tau,t) = \int_{\tau^0}^{\tau} D_c U(s,\tau,t)\cdot v(s,h(s,\tau,t))ds + v(\tau,t) =$$

$$= A(\tau,t)\int_{\tau^0}^{\tau} U(s,\tau,t)v(s,h(s,\tau,t))ds +$$

$$+ \int_{\tau^0}^{\tau}\left(\int_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))U(s,\xi,h(\xi,\tau,t))d\xi\right)v(s,h(s,\tau,t))ds + v(\tau,t) =$$

$$= A(\tau,t)u^*(\tau^0,\tau,t) +$$

$$+ \int_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))\left\{\int_{\tau^0}^{\tau} U(s,\xi,h(\xi,\tau,t))v(s,h(s,\xi,h(\xi,\tau,t)))ds\right\}d\xi + v(\tau,t) =$$

$$= A(\tau,t)u^*(\tau^0,\tau,t) + \int_{\tau-\varepsilon}^{\tau} K(\tau,t,\xi,h(\xi,\tau,t))u^*(\tau^0,\xi,h(\xi,\tau,t))d\xi + v(\tau,t). \quad (4.5)$$

Substituting expressions (4.3) and (4.5) into system (4.1) we obtain that

$$v(\tau,t) = f(\tau,t). \quad (4.6)$$

Then, by virtue of (4.6), we find the desired solution

$$u^*\left(\tau^0,\tau,t\right)=\int_{\tau^0}^{\tau}U(s,\tau,t)f(s,h(s,\tau,t))ds\,. \qquad (4.7)$$

Obviously, the solution (4.7) satisfies condition (4.1^*).

Since the solution $u\left(\tau^0,\tau,t\right)$ of the linear inhomogeneous equation consists of the sum of the solutions of the homogeneous equation and some particular solution of the inhomogeneous equation, we have the expression for the general Cauchy solution

$$u\left(\tau^0,\tau,t\right)=U\left(\tau^0,\tau,t\right)u^0\left(h\left(\tau^0,\tau,t\right)\right)+u^*\left(\tau^0,\tau,t\right) \qquad (4.8)$$

system (4.1) with initial condition (4.1^0).

Thus, we have the theorem on solving the initial problem for a linear inhomogeneous system of integro-differential equations (4.1).

Theorem 4.1. *Under conditions (3.2), (3.3) and (4.2), the initial problem (4.1) - (4.1^0) has the unique solution in the form (4.7) - (4.8).*

◄ The existence of a solution $u\left(\tau^0,\tau,t\right)$ under the conditions of the theorem is proved by the deductions of formulas (4.7) and (4.8). The uniqueness of the solution (4.8) follows from the uniqueness of solution of the initial problem for the homogeneous system (3.1).►

Now we will research the problems of multiperiodic solutions of system (4.1). Suppose that the conditions of theorem 3.7 are satisfied.

Then the homogeneous system (3.1) corresponding to system (4.1) does not have (θ,ω)-periodic non-zero solutions, and it has the property of exponential dichotomy.

In this case, the question of the existence of multiperiodic solutions of system (4.1) is investigated on the basis of the Green's function method.

To construct the Green's function $G(s,\tau,t)$, using the property of exponential dichotomy, we set

$$G(s,\tau,t)=\begin{cases}U_-(s,\tau,t), & \tau\geq s,\\ -U_+(s,\tau,t), & \tau<s,\end{cases} \qquad (4.9)$$

where $U_-(s,\tau,t)$ and $U_+(s,\tau,t)$ are the terms of the resolving operator $U(s,\tau,t)$, which consists of their sum (3.35).

The constructed Green function (4.9) possess the following properties.

$$1^0.\, D_cG(s,\tau,t)=A(\tau,t)G(s,\tau,t)+ + \int_{\tau-\varepsilon}^{\tau}K(\tau,t,\xi,h(\xi,\tau,t))G(s,\xi,h(\xi,\tau,t))d\xi,\tau\neq s\,. \qquad (4.10)$$

This property follows from property (3.36) of the operator $U(s,\tau,t)$.

$$2^0.\; G(s-0,\tau,t)-G(s+0,\tau,t)=E\,. \qquad (4.11)$$

Lets note that follows from the equality $U_-(\tau-0,\tau,t)+U(\tau+0,\tau,t)=$ $=U_-(\tau,\tau,t)+U_+(\tau,\tau,t)=U(\tau,\tau,t)=E$.

$$3^0.\; G(s+\theta,\tau+\theta,t+q\omega)=G(s,\tau,t),\; q\in Z^m\,. \qquad (4.12)$$

This property is the consequence of property (3.17) of the resolving operator $U(s,\tau,t)$.

$$4^0.\,|G(s,\tau,t)|\leq a\,e^{-\alpha|\tau-s|},\; a\geq 1 \text{ and } \alpha>0\,. \qquad (4.13)$$

We have this estimate from inequalities (3.37) - (3.38).

Theorem 4.2. *Suppose that the conditions of theorem 4.1 are satisfied and the matrix* $A(\tau, t)$ *with kernel* $K(\tau, t, s, \sigma)$ *are such that the linear homogeneous system (3.1) has the property of exponential dichotomy, expressed by the relation (3.35) - (3.38). Then system (4.1) has the unique* (θ, ω)*-periodic solution*

$$u^*(\tau, t) = \int_{-\infty}^{+\infty} G(s, \tau, t) f(s, h(s, \tau, t)) ds, \tag{4.14}$$

satisfying estimate

$$\|u^*\| \le \frac{a}{\alpha}\|f\|, \tag{4.15}$$

where $\|u\| = \sup_{R \times R^m} |u(\tau, t)|$.

◄ The convergence of the integral (4.14) and the differentiability of the solution (4.14) follow from the differentiability of the matrix-vector functions G, f and estimate (4.13). By virtue of (4.10) and (4.11), it is proved that the vector-function (4.14) satisfies system (4.1). Multiperiodicity follows from properties (4.12) and (4.2). Inequality (4.15) is the consequence of the estimate (4.13). The exponential dichotomy of system (3.1) ensures the uniqueness of the (θ, ω)-periodic solution of system (4.1). ►

Lemma 4.1. *Let the homogeneous linear system (3.1) under conditions (3.2), (3.3) and (4.2) have no* (θ, ω)*-periodic solutions except zero. Then the corresponding inhomogeneous linear system (4.1) can have at most one* (θ, ω)*-periodic solution.*

◄ Suppose that under the conditions of this lemma, system (4.1) has two different (θ, ω)-periodic solutions $u_1(\tau, t)$ and $u_2(\tau, t) \ne u_1(\tau, t)$. Then their difference $u(\tau, t) = u_2(\tau, t) - u_1(\tau, t) \ne 0$ is a (θ, ω)-periodic solution of the linear homogeneous system (3.1), which has only a zero (θ, ω)-periodic solution. The obtained contradiction proves the validity of the lemma. ►

Assuming that the ω-periodic initial function $u^0(t) \in C_t^{(e)}(R^m)$ of the (θ, ω)-periodic solution $u(\tau, t)$:

$$u(\tau, t) = U(0, \tau, t) u^0(h(0, \tau, t)) + u^*(0, \tau, t), \tag{4.16}$$

represented by (4.8) has property

$$u^0(t - c\theta) = u^0(t), \tag{4.17}$$

it can also be represented by formula

$$u(\tau, t) \equiv u(\tau + \theta, t) = U(0, \tau + \theta, t) u^0(h(0, \tau, t)) + u^*(0, \tau + \theta, t), \tag{4.18}$$

since, by condition (4.17) $u^0(h(0, \tau, t))$ is (θ, ω)-periodic zero operator D_c.

Then, eliminating the unknown initial function $u^0(t)$ from relations (4.16) and (4.18), we obtain

$$u(\tau, t) = \left[U^{-1}(0, \tau + \theta, t) - U^{-1}(0, \tau, t)\right] \times$$
$$\times \left\{U^{-1}(0, \tau + \theta, t) u^*(0, \tau + \theta, t) - U^{-1}(0, \tau, t) u^*(0, \tau, t)\right\}. \tag{4.19}$$

Further, using representation (4.7) of the solution $u^*(0, \tau, t)$, accepting the notation $\widetilde{U}(s, \tau, t) = U^{-1}(0, \tau, t) U(s, \tau, t)$ and setting

$$\widetilde{U}_\theta(s,\tau,t)=\begin{cases}\widetilde{U}(s,\tau,t), & \tau \xrightarrow{s} 0,\\ \widetilde{U}(s,\tau+\theta,t), & 0 \xrightarrow{s} \tau+\theta,\end{cases} \tag{4.20}$$

$$f_\theta(s,\tau,h(s,\tau,t))=\begin{cases} f(s,\tau,h(s,\tau,t)), & \tau \xrightarrow{s} 0,\\ f(s,\tau+\theta,h(s,\tau+\theta,t)), & 0 \xrightarrow{s} \tau+\theta,\end{cases} \tag{4.21}$$

we can represent relation (4.19) in the form

$$u(\tau,t)=\left[U^{-1}(0,\tau+\theta,t)-U^{-1}(0,\tau,t)\right]^{-1}\int_{\tau}^{\tau+\theta}\widetilde{U}_\theta(s,\tau,t)f_\theta(s,h(s,\tau,t))ds\,. \tag{4.22}$$

Thus, looking for a (θ,ω)-periodic solution of system (4.1) among solutions $u(\tau,t)$ with initial conditions having property (4.17), we showed that it is determined by formula (4.22), which is revealed by relations (4.7) and (4.19) - (4.21).

Theorem 4.3. *Suppose that conditions (3.2), (3.3), (4.2) are satisfied and the linear homogeneous system (3.1) has no (θ,ω)-periodic solutions, except for the trivial one. Then the system of inhomogeneous linear integro-differential equations (4.1) has the unique (θ,ω)-periodic solution $u(\tau,t)$ of the form (4.22).*

◄ The conclusion of the solution (4.22) is given above. Therefore, the existence of the (θ,ω)-periodic solution of system (4.1) is proved. Uniqueness follows from Lemma 4.1. ►

Note that the above researched problems for the considered systems can be considered along the characteristics $t=t^0+c\tau-c\tau^0$ with fixed initial data (τ^0,t^0).

Then, due to the fact that the operator D_c turns into the operator $d/d\tau$ of the full derivative, from the theorems proved as a corollary we have the corresponding statements about the existence of solutions to the initial problems for systems of ordinary integro-differential equations and the theorem about the existence of their Bohr quasiperiodic solutions generated by multiperiodic solutions of the original systems which we will not dwell on here.

Conclusion.

First of all, we've noted that this article proposes the method for (research) researching solutions of problems that satisfy the initial conditions and have the property of multiperiodicity with given periods for linear systems of integro-differential equations with D_c partial differential operator, ε-hereditary effect and the linear integral operator. This technique is a generalization of methods and solutions of similar problems for systems of partial differential equations with the operator D_c. The solution of problems under consideration for such systems in this formulation are researched for the first time. The relevance of the main problem is substantiated. The solutions of all the subtasks analyzed to achieve the goal are formulated as theorems with proofs. Scientific novelties include the multi-periodicity theorems of zeros of the operator D_c; about solutions to initial problems for all considered types of systems; about necessary as well as sufficient conditions for the existence of multiperiodic solutions of both homogeneous and inhomogeneous systems, as well as the integral representations of solutions inhomogeneous systems in two cases when the corresponding homogeneous systems have the properties: 1) exponential dichotomy and 2) the absence of non-trivial multiperiodic solutions, at all.

We've also noted that the consequences deduced by examining the results obtained along the characteristics $t=t^0+c\tau-c\tau^0$ with fixed (τ^0,t^0) refer to their applications in the theory of quasiperiodic solutions of systems ordinary integro-differential equations.

The technique that developed here is quite applicable to the research of problems of hereditary-string vibrations and the "predator-prey" given in the delivered part of the work, which can be attributed to examples of the applied aspect.

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D_c-ОПЕРАТОРЛЫ ЖӘНЕ ε-ЭРЕДИТАРЛЫҚ ПЕРИОДТЫ СЫЗЫҚТЫ ИНТЕГРАЛДЫ-ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІНІҢ КӨППЕРИОДТЫ ШЕШІМДЕРІ

Аннотация. Мақалада $D_c = \partial/\partial\tau + c_1\,\partial/\partial t_1 + \ldots + c_m\,\partial/\partial t_m$ операторлы, $c = (c_1, \ldots, c_m) - const$ және тұқым қуалаушылық сипаттағы құбылыстарды сипаттайтын ε ақырлы эредитарлық периодты сызықты интегралды-дифференциалдық теңдеулер жүйесінің көппериодты шешімдері жөніндегі есептер мен бастапқы есеп мәселелері зерттеледі. D_c операторының нөлдерінің теңдеуімен қатар сызықты біртекті және біртекті емес интегралды-дифференциалдық теңдеулер жүйесі қарастырылды, олар үшін бастапқы есептердің бірмәнді шешілімділігінің жеткілікті шарттары анықталған, (τ, t) бойынша (θ, ω) периодты, көппериодты шешімдердің бар болуының қажетті де, жеткілікті де шарттары алынған. Сызықты біртекті емес жүйенің көппериодты шешімдерінің интегралдық өрнектері 1) дербес жағдайда, яки теңдеуге сәйкес біртекті жүйелер экспоненциалды дихотомиялық қасиетке ие болғанда және 2) жалпы жағдайда, біртекті жүйелердің нөлден басқа көппериодты шешімдері болмағанда айқындалды.

Түйін сөздер: интегралды-дифференциалдық теңдеу, эредитарлық, флуктуация, көппериодты шешім.

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МНОГОПЕРИОДИЧЕСКИЕ РЕШЕНИЯ ЛИНЕЙНЫХ СИСТЕМ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С D_c-ОПЕРАТОРОМ И ε-ПЕРИОДОМ ЭРЕДИТАРНОСТИ

Аннотация. В заметке исследуются вопросы начальной задачи и задачи о многопериодичности решений линейных систем интегро-дифференциальных уравнений с оператором вида $D_c = \partial/\partial\tau + c_1\,\partial/\partial t_1 + \ldots + c_m\,\partial/\partial t_m$, $c = (c_1, \ldots, c_m) - const$ и конечным периодом эредитарности $\varepsilon = const > 0$, которые описывают явления наследственного характера. Наряду с уравнением нулей оператора D_c рассмотрены линейные системы однородных и неоднородных интегро-дифференциальных уравнений, для них установлены достаточные условия однозначной разрешимости начальных задач, получены как необходимые, так и достаточные условия существования многопериодических по (τ, t) с периодами (θ, ω) решений. Определены интегральные представления многопериодических решений линейных неоднородных систем 1) в частном случае, когда соответствующие однородные системы обладают экспоненциальной дихотомичностью и 2) в общем случае, когда однородные системы не имеют многопериодических решений, кроме тривиального.

Ключевые слова: интегро-дифференциальное уравнение, эредитарность, флуктуация, многопериодическое решение.

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REFERENCES

[1] Volterra V. **(1912)** Sur les equations integro-differentielles et leurs applications. Acta Math., 35, 295-356.
[2] Volterra V. **(1912)** Vibrazioni elastiche nel caso della eredita. Rend. R. Accad. Dei Lincel. (5)21, 3-12.
[3] Volterra V. **(1913)** Lecons sur les equations integrals et les equations integro-differentielles. Paris.
[4] Evans G.C. **(1912)** Sull equazione integro-dirrerenziale di tipo parabolico. – Rend.R. Accad. Dei Lincei. (5)21, 25-31.

[5] Volterra V. **(1976)** The mathematical theory of the struggle for existence [Matematicheskaja teorija borby za sushhestvovanie]. Moscow, Nauka (in Russian).

[6] Volterra V. **(1982)** Theory of functionals, integral and integro-differential equations [Teorija funkcionalov, integral'nyh i integro-differencial'nyh uravnenij]. Moscow, Nauka (in Russian).

[7] Graffi D. **(1928)** Sulla theoria delle jscillazioni elastiche con ereditarieta. Nuovo Cimento. (2)5, 310-317.

[8] Bykov Ia.V. **(1957)** On some problems of the theory of integro-differential equations [O nekotoryh zadachah teorii integro-differencial'nyh uravnenij]. Frunze, Kirgiz. gos. un-t. (in Russian).

[9] Imanaliev M.I. **(1974)** Oscillations and stability of solutions of singularly perturbed integro-differential systems [Kolebanija i ustojchivost' reshenij singuljarno-vozmushhennyh integro-differencial'nyh sistem]. Frunze, Ilim (in Russian).

[10] Imanaliev M.I. **(1992)** Partial nonlinear integro-differential equations [Nelinejnye integro-differencial'nye uravnenija s chastnymi proizvodnymi]. Bishkek, Ilim (in Russian).

[11] Barbashin E.A. **(1967)** Introduction to sustainability theory [Vvedenie v teoriju ustojchivosti]. Moscow, Nauka (in Russian).

[12] Samoilenko A.M., Nurzhanov O.D. **(1979)** The Bubnov-Galerkin method of constructing periodic solutions of Voltaire type integro-differential equations [Metod Bubnova-Galerkina postroenija periodicheskih reshenij integro-differencial'nyh uravnenij tipa Vol'tera]. Diffrents. Uravneniia - Diffrenz. the equations, 15: 8, 1503-1517 (in Russian).

[13] Botashev A.I. **(1998)** Periodic solutions of Volterra integro-differential equations [Periodicheskie reshenija integro-differencial'nyh uravnenij Vol'terra]. Moscow, Izd-vo MFTI - MIPT Publishing House (in Russian).

[14] Nakhushev A.M. **(1995)** Equations of mathematical biology [Uravneniia matematicheskoi biologii]. Moscow, Vysshaia shkola (in Russian).

[15] Rabotnov Iu.N. **(1977)** Elements of hereditary solid mechanics [Elementy nasledstvennoi mekhaniki tverdykh tel]. Moscow, Nauka (in Russian).

[16] Iliushin A.A., Pobedria B.E. **(1970)** Fundamentals of the mathematical theory of thermo-viscous-elasticity [Osnovy matematicheskoi teorii termoviazko-uprugosti]. Moscow, Nauka (in Russian).

[17] Dzhumabaev D.S., Bakirova E.A., Kadirbayeva Zh.M. **(2018)** An algorithm for solving a control problem for a differential equation with a parameter. News of NAS RK. Series of physico-mathematical. Volume 5, Number 321 (2018), PP. 25 – 32. https://doi.org/10.32014/2018.2518-1726.4

[18] Assanova A.T., Alikhanova B.Zh., Nazarova K.Zh. **(2018)** Well-posedness of a nonlocal problem with integral conditions for third order system of the partial differential equations. News of NAS RK. Series of physico-mathematical. Volume 5, Number 321 (2018), PP. 33 – 41. https://doi.org/10.32014/2018.2518-1726.5

[19] Kharasakhal V.Kh. **(1970)** Almost periodic solutions of ordinary differential equations [Pochti periodicheskie resheniia obyknovennykh differentsialnykh uravnenii]. Alma-Ata, Nauka (in Russian).

[20] Umbetzhanov D.U. **(1979)** Almost periodic solutions of partial differential equations [Pochti periodicheskie reshenija differencial'nyh uravnenij v chastnyh proizvodnyh]. Alma-Ata, Nauka (in Russian).

[21] Umbetzhanov D.U. **(1990)** Almost periodic solutions of evolutionary equations [Pochti periodicheskie resheniia evoliutsionnykh uravnenii]. Alma-Ata, Nauka (in Russian).

[22] Umbetzhanov D.U. **(1989)** On an almost multiperiodic solution of one integro-differential equation of transport type [O pochti mnogoperiodicheskom reshenii odnogo integro-differencial'nogo uravnenija tipa perenosa]. Ukr. Math. Journal, №1. 79-85. (in Russian).

[23] Umbetzhanov D.U., Sartabanov Zh.A. **(1977)** On an almost-multiperiodic solution of a countable system of partial differential integro-differential equations [O pochti- mnogoperiodicheskom reshenii odnoj schetnoj sistemy integro-differencial'nyh uravnenij v chastnyh proizvodnyh]. Mathematics and Applied Mathematics. Matvuz. KazSSR, Alma-Ata, 102-108 (in Russian).

[24] Umbetzhanov D.U., Berzhanov, A.B. **(1983)** On the existence of an almost multi-periodic solution of one system of integro-differential partial differential equations [O sushchestvovanii pochti mnogoperiodicheskogo resheniia odnoi sistemy integro-differentsialnykh uravnenii v chastnykh proizvodnykh]. Izv. AN KazSSR. Ser. fiz.-mat. - Izv. Academy of Sciences of the Kazakh SSR. Ser. Phys. Mat., 5, 11-15 (in Russian).

[25] Sartabanov Zh.A. **(1987)** Multiperiod solution of one system of integro-differential equations [Mnogoperiodicheskoe reshenie odnoj sistemy integro-differencial'nyh uravnenij]. Izv. AN KazSSR. Ser. fiz.-mat. - Izv. Academy of Sciences of the Kazakh SSR. Ser. Phys. Mat., 5, 51-56 (in Russian).

[26] Sartabanov Zh.A. **(1989)** Pseudoperiodic solutions of one system of integro-differential equations. Ukr. Math. Journal, 41:1. 116-120. DOI:10.1007BF01060661.

[27] Sartabanov Zh.A. **(2001)** Periodic functions and periodic solutions of some ordinary differential equations. Almaty, RBK - RBC (in Kazakh).

[28] Mukhambetova A.A., Sartabanov Zh.A. **(2007)** Stability of solutions of systems of differential equations with multidimensional time [Ustojchivost' reshenij sistem differencial'nyh uravnenij s mnogomernym vremenem] Aktobe, Print A (in Russian).

[29] Kulzhumieva A.A., Sartabanov Zh.A. **(2013)** Periodic solutions of systems of differential equations with multidimensional time [Periodicheskie reshenija sistem differencial'nyh uravnenij s mnogomernym vremenem]. Uralsk. ZKGU, RIC (in Russian). ISBN 978-601-266-128-6.

[30] Abdikalikova G.A. **(2001)** Construction of an almost periodic solution of one quasilinear parabolic system [Postroenie pochti periodicheskogo reshenija odnoj kvazilinejnoj parabolicheskoj sistemy]. Izv. MON NAN RK. Ser.fiz.-mat., - Izv. MON NAS RK. Ser. Phys. Mat., 3, 3-8 (in Russian).

[31] Kulzhumiyeva A.A., Sartabanov Zh.A. **(2017)** On multiperiodic integrals of a linear system with the differentiation operator in the direction of the main diagonal in the space of independent variables. Eurasian Mathematical Journal. 8:1, 67-75.

[32] Sartabanov Zh.A. **(2017)** The multi-periodic solutions of a linear system of equations with the operator of differentiation along the main diagonal of the space of independent variables and delayed arguments. AIP Conference Proceedings 1880, 040020.

[33] Kulzhumiyeva A.A., Sartabanov Zh.A. **(2019)** Integration of a linear equation with differential operator, corresponding to the main diagonal in the space of independent variables, and coefficients, constant on the diagonal. Russian Mathematics. 63:6, 29-41. DOI:10.3103S1066369X19060045.

[34] Sartabanov Zh.A., Omarova B.Z. **(2018)** Multiperiodic solutions of autonomous systems with operator of differentiation on the Lyapunov's vector field. AIP Conference Proceedings 1997, 020041 (2018). https://doi.org/10.1063/1.5049035

[35] Abdikalikova G.A., Aitenova G.M., Sartabanov Zh.A. **(2018)** Multiperiodic solution of a system of integro-differential equations in partial derivatives [Mnogoperiodicheskoe reshenie sistemy integro-differencial'nyh uravnenij v chastnyh proizvodnyh]. Vestnik KazNPU im.Abaja. Ser. «fiz.-mat.» - Bulletin of KazNPU named after Abay. Ser. "Phys.-Math.". 63:3, 5-11 (in Russian).

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ON THE APPLICATION OF QUADRATURE FORMULAS FOR CALCULATING INTEGRALS OF ARBITRARY MULTIPLICITY

Abstract. In this paper, we consider the calculation of integrals of arbitrary multiplicity by the methods of non-uniform grids, Monte Carlo and optimal coefficients. A comparative analysis of these numerical methods for integrating multiple integrals was made. It was established that the method of optimal coefficients had an advantage compared to other methods. It is shown that the use of uneven and parallelepipedal grids is the basis of almost all results obtained in the field of application of theoretic – numerical methods to the problems of approximate analysis. It is established that interpolation of functions of several variables by theoretic-numerical grids allows to receive interpolation formulas, accuracy of which rises with increase of smoothness of functions. The number of variables in this case has no significant effect on the order of the residual member. The use of the function $f \in E_s^\alpha$ to Fourier coefficients allows to obtain an interpolation formula from the quadrature formulas, which are constructed with the help of the parallelepipedal grids. This formula is accurate for trigonometric polynomials, the degree of which does not exceed the value of $\sqrt{N} \ln^{-\frac{s}{2}} N$.

Key words: theoretic-numerical method, quadrature formula, method of optimal coefficients, multiple integrals.

1. Introduction

There are three types of problems in which theoretic – numerical approaches lead to general results: application of quadrature formulas for calculating integrals of arbitrary multiplicity, approximate solution of integral equations and interpolation of functions of several variables.

The paper considers the connection between the theory of uniform distribution and the number-theoretic method in approximate analysis. The main types of theoretic – numerical grids, non-uniform and parallelepipedal, are analyzed. The problems of finding the optimal coefficients for parallelepipedal grids are presented.

Theoretic – numerical algorithms of numerical integration are essential in the calculation of interaction integrals in quantum chemistry and in the calculation of nanoscale ferromagnetic heterosystems, and also in high-energy physics.

2. Materials and methods of research

2.1 Approximate calculation of multiple integrals

Integration of multiple integrals of functions of the class E_s^α.

The function of the form:

$$f(x_1,...,x_s) = \sum_{m_1,...,m_s=-\infty}^{\infty} C(m_1,...,m_s) e^{2\pi i(m_1x_1+...m_sx_s)} \tag{1}$$

belongs to the class E_s^α if $C(m_1,..., m_s) = O\left(\left(\overline{m_1},..., \overline{m_s}\right)^{-\alpha}\right)$, where $\overline{m_\nu} = \max(1, |m_\nu|)$ and the value of $\alpha > 1$ characterizes the smoothness of functions.

The class E_s^α has periodic functions that have continuous derivatives of the form $\frac{\partial^{\alpha s} f}{\partial x_1^{v_1} \ldots \partial x_s^{v_s}}$, where $v_1,\ldots,v_s$ is an arbitrary permutation of fixed integers $\alpha_1,\ldots,\alpha_s$ selected from the interval $[0,\alpha s]$, so that $\alpha_1+\ldots+\alpha_s=\alpha s$. In particular, for integer α, functions that have a derivative $\frac{\partial^{\alpha s} f}{\partial x_1^{\alpha} \ldots \partial x_s^{\alpha}}$ will belong to the class E_s^α.

It is not necessary for the function to be periodic. There are simple ways of transition from non-periodic functions to periodic functions [1-3]. The replacement of variables that do not disrupt the smoothness of the functions and does not lead to significant complication of calculations can also be used for the periodization of the function.

Let R be the error of the simplest quadrature formula

$$\int_0^1 \cdots \int_0^1 f(x_1,\ldots,x_s)dx_1\ldots dx_s = \frac{1}{N}\sum_{k=1}^{N} f[\xi_1(k),\ldots,\xi_s(k)] - R, \tag{2}$$

where the collection of points $M_k=[\xi_1(k),\ldots,\xi_s(k)]$ is called a grid.

In the case of uniform grids, arising from partition of the unit s-dimensional cube into $N=n^s$ equal small cubes, the following estimate is valid for the functions of the class E_s^α:

$$R = O\left(\frac{1}{N^{\frac{\alpha}{s}}}\right), \tag{3}$$

achievable in this class; this estimate is not improved when using quadrature formulas with arbitrary weights. The disadvantage of quadrature formulas with uniform grids is the decrease in their accuracy with the increase of the number of measurements.

2.2 The first theoretic – numerical method for constructing quadrature formulas

This method is based on the use of non-uniform grids of the form [1]

$$M_k = \left(\left\{\frac{k}{N}\right\},\ldots,\left\{\frac{k^s}{N}\right\}\right), \tag{4}$$

where N is the prime number, $\left\{\frac{k^v}{N}\right\}$ is the fractional proportion of the number $\frac{k^v}{N}$.

In the case of non-uniform grids, the error estimate of the quadrature formula (2) takes the form

$$R = O\left(\frac{1}{\sqrt{N}}\right). \tag{5}$$

Non-uniform grids are obviously free from the main drawback of uniform grids – unlike estimate (3) the order of estimate (5) remains unchanged with the increase of the number of measurements.

Along with the above-mentioned advantage of non-uniform grids, these grids have a significant disadvantage – the accuracy of the results obtained using the corresponding quadrature formulas does not increase with increasing smoothness of the considered functions.

2.3 The second theoretic – numerical method for constructing quadrature formulas

This method is based on the use of parallelepipedal grids of the form

$$M_k = \left(\left\{\frac{ka_1}{N}\right\},\ldots,\left\{\frac{ka_s}{N}\right\}\right), \tag{6}$$

where $a_1,\ldots,a_s$ are integer numbers selected in a special way (optimal coefficients). This method does not have the disadvantage of non-uniform grids [5]. For parallelepipedal grids, the error estimate in formula (2) takes the form

$$R = O\left(\frac{\ln^{\alpha s} N}{N^{\alpha}}\right). \qquad (7)$$

There are no grids that give better estimate R on the class E_s^{α} than $O\left(\frac{1}{N^{\alpha}}\right)$ [6]. This estimate cannot be improved for the case of quadrature formulas of the most general form as well [7]. Thus, grids of the form (6) lead to quadrature formulas, in which the error estimate does not allow further significant improvement.

When $s = 1$, there exists a single parallelepipedal grid, which coincides with the uniform grid, obtained by dividing the segment $[0,1]$ into N equal parts, i.e.

$$M_k = \left(\left\{\frac{k}{N}\right\}\right), \text{ where } k = 1,2,...,N \qquad (8)$$

When $s = 2$, it is not difficult to show that the integers $a_1 = 1$, $a_2 = a$ will be optimal coefficients for any $a = a(N)$, in which incomplete partial relations $\frac{a}{N}$ will be limited to a value increasing with the growth of N not more than some degree of $\ln N$. In particular, when $N = u_n$, where u_n is a general term of the Fibonacci sequence (the Fibonacci sequence is defined as follows: $u_0 = 1$, $u_1 = 1$, $u_n = u_{n-1} + u_{n-2}$, $n = 2,3,...$), the integers $a_1 = 1$, $a_2 = u_{n-1}$ will be optimal coefficients and points

$$M_k = \left(\left\{\frac{k}{u_n}\right\}, \left\{\frac{ku_{n-1}}{u_n}\right\}\right), \text{ where } k = 1,2,...,u_n \qquad (9)$$

form a two-dimensional parallelepipedal grid.

When $s \geq 3$, various sufficient optimality conditions can be used to calculate the optimal coefficients.

Let at $v = 1,2,...,s$ for integers z_v from the segment $[1, N-1]$ the functions $H(z_1,...,z_v)$ are determined by the equality [4]:

$$H(z_1,...,z_v) = \sum_{k=1}^{N-1}\left[1 - 2\ln\left(2\sin\pi\left\{\frac{kz_1}{N}\right\}\right)\right]\cdots\left[1 - 2\ln\left(2\sin\pi\left\{\frac{kz_s}{N}\right\}\right)\right].$$

The integers $a_1,...,a_s$ will be optimal coefficients if $a_1 = 1$ and for given $a_1,...,a_{v-1}$ ($v \geq 2$) the value a_v is equal to any of the values z_v, at which the minimum of the function $H(a_1,...,a_{v-1},z_v)$ is reached.

Another sufficient condition for optimality according to [1] – the integers $a_1,...,a_s$ are optimal coefficients if the minimum multiplication $\overline{m_1},\cdots,\overline{m_s}$ for non-trivial solutions of the comparison $a_1m_1 + ... + a_sm_s \equiv 0(\bmod N)$ satisfies the condition $\overline{m_1},\cdots,\overline{m_s} > BN\ln^{-\gamma} N$, where $B > 0$ and $\gamma \geq 0$ are constants depending only on s.

From relations (4), (8) and (9) it can be seen that non-uniform grids with any s and parallelepipedal grids with $s \leq 2$ are indicated quite effectively with the help of simple formulas. When $s \geq 3$, different algorithms have to be used to find parallelepipedal grids. Consideration of algorithms, in which the number of operations necessary to specify the grid is not too large compared to the number of calculations in the corresponding quadrature formulas, can be practically effective.

The first of the above methods for finding optimal coefficients is practically effective, since the number of elementary operations in the calculations arising in it has order N^2. By slightly modifying this algorithm, it is possible to reduce the number of operations to $O(N^{1+\varepsilon})$, where $\varepsilon > 0$ is arbitrarily small.

Almost all the results obtained using parallelepipedal grids, it is possible to use practically effective algorithms. However, in some cases [1], the indication of the corresponding grids is still possible only with the help of $O(N^s)$ operations, where $s > 2$ is ineffective.

From the two grids leading to the following error estimates $|R| < C_1 N^{-\alpha_1} \ln^{\beta_1} N$ and $|R| < C_2 N^{-\alpha_2} \ln^{\beta_2} N$, it is natural to consider the first one, which is better if $\alpha_1 > \alpha_2$. However, the advantage of the first grid can be revealed only at very large values of N. Therefore, in computational practice, when choosing a quadrature formula, appropriate experiments are necessary.

3 Comparison of numerical methods for integrating multiple integrals

The multiple integral over the unit volume of some function $f(x_1, x_2, x_3, x_4)$ is replaced by the finite sum:

$$\int_0^1\int_0^1\int_0^1\int_0^1 f(x_1,x_2,x_3,x_4)dx_1dx_2dx_3dx_4 = \frac{1}{pq}\sum_{k=1}^{pq} f\left(\left\{\frac{a_0k}{pq}\right\},\left\{\frac{a_1k}{pq}\right\},\left\{\frac{a_2k}{pq}\right\},\left\{\frac{a_3k}{pq}\right\}\right),$$

where $pq = 4097$, and a_0, a_1, a_2, a_3 are optimal coefficients. The integrals of the following functions are calculated:

1. $\dfrac{x_1x_2x_3x_4}{0.0625}$; 2. $\dfrac{x_1 + x_2 - x_3 + 2x_4}{1.5}$; 3. $\dfrac{x_1^3 x_2^2 x_3 e^{x_1x_2x_3x_4}}{0.051615162}$;

4. $1 + \cos 2\pi(x_1 + x_2 + x_3 + x_4)$;

5. $1 + \sin a\pi(x_1 + x_2 - x_3 + 2x_4)$ for $a = 10;30;60;100;200$;

6. $\dfrac{\prod_{i=1}^{4}\left(\dfrac{1}{1-x_i}\right)^2 e^{-a\left(\frac{x_i}{1-x_i}\right)^2}}{\dfrac{\pi^2}{16a^2}}$ for $a = 1;10;30;60;100.$

In integrals, non-periodic integrands were periodized and the integrals of them are equal to 1.

The following table shows the results of calculating the integrals in different ways. It shows that the method of optimal coefficients has an advantage over the calculation by other methods. And also in most cases, a significant advantage of parallelepipedal grids over other grids is revealed even with a very small value of N [15].

Table - Results of calculating integrals by different methods

Functions		Method of non-uniform grids	Monte Carlo methods				Method of optimal coefficients
			1	2	3	4	
1		0.38	0.78	1.05	1.02	1.01	0.999995
2		0.74	0.94	0.99	1.008	1.008	0.999999
3		0.80	0.75	1.07	1.04	0.99	1.000186
4		1.01	1.009	1.002	0.99	1.01	1.000000
5	10	1.01	1.008	0.99	0.99	0.99	1.00000000
	30	1.01	0.98	0.99	0.99	0.99	1.00000002
	60	1.01	0.98	0.99	1.008	1.003	1.00000004
	100	1.02	1.02	1.003	0.99	0.99	1.00000013
	200	0.98	1.01	0.99	1.0001	1.02	1.00000021
6	1	1.03	1.23	1.03	0.93	0.99	0.999682
	10	1.39	1.65	1.13	0.84	0.77	1.002806
	30	3.22	2.48	0.87	0.91	0.95	0.940240
	60	7.48	3.87	0.53	0.54	1.16	1.583021
	100	16.2	5.99	0.22	0.19	1.28	3.977712

All the obtained sets of optimal coefficients and the values of five, ten, fifteen-fold integrals calculated by the method of optimal coefficients with parallelepipedal grids for non-periodic functions are given in [8-14]. It was shown that the calculation of integrals with good accuracy was also possible for the small number of nodes of the quadrature formula N. A comparative characteristic of the calculated sets of optimal coefficients and values of multiple integrals by the theoretic - numerical methods, taking into account the number of nodes of the quadrature formula, was also given.

The use of non-uniform and parallelepipedal grids forms the basis of almost all the results obtained in the field of application of theoretic – numerical methods to the problems of approximate analysis.

For a multiple Fredholm integral equation of 2nd kind:

$$\varphi(P) = \lambda \int_{G_s} k(P,Q)\varphi(Q)dQ + f(P), \tag{10}$$

where integration is extended to a unit s – dimensional cube G_s. We will assume that the free term and the core of this equation belong to the classes E_s^α and E_{2s}^α respectively, and that the denominator of Fredholm $D(\lambda)$ is non – zero. Using theoretic - numerical grids M_k, one can obtain [5] an approximate solution of equation (10) in the form

$$\varphi(P) = \frac{\lambda}{N}\sum_{k=1}^{N} K(P, M_k)\tilde{\varphi}(M_k) + f(P) + R,$$

where values of $\tilde{\varphi}(M_k)$ are determined from a system of linear algebraic equations:

$$\tilde{\varphi}(M_k) = \frac{\lambda}{N}\sum_{l=1}^{N} K(M_k, M_l)\tilde{\varphi}(M_l) + f(M_k), \text{ where } k = 1,2,\ldots,N;$$

moreover, the error R, depending on the choice of grids, has the same order as in the calculation of multiple integrals of functions belonging to the class E_s^α.

For an arbitrarily small $\varepsilon > 0$ and sufficiently small λ, using the method of iterations and non-uniform grids of the form (4) to calculate the increasing multiplicity integrals, we can obtain an explicit approximate expression for $\varphi(P)$:

$$\varphi(P) = f(P) + \frac{1}{N}\sum_{k=1}^{N}\sum_{v=1}^{n}\lambda^v K(P, M_{1,k})\ldots K(M_{v-1,k}, M_{v,k}) f(M_{v,k}) + O\left(N^{-\frac{1}{2}+\varepsilon}\right).$$

Here $M_{v,k} = \left(\left\{\frac{k^{s(v-1)+1}}{N}\right\},\ldots,\left\{\frac{k^{sv}}{N}\right\}\right)$, $n = [y\ln N]$ is the integer part of the quantity $y\ln N$ and y is some constant depending on ε and the character of decreasing Fourier coefficients of the kernel of equation (10).

Using parallelepipedal grids and slightly changing the definition of classes E_s^α [6], it is possible in the analytical expression for $\varphi(P)$ to improve the residual term to $O(N^{-\alpha+\varepsilon})$. The same methods can be applied [15] to solving multiple Volterr equations and equations of the mixed type, in which some of the integrations are constant and some of them are in variable limits. In [16], questions of the numerical solution of nonlinear Volterr integral equations of the first kind with a differentiable kernel, which degenerates at the initial point of the diagonal, are considered. It is shown that this equation reduces to the Volterr integral equation of the third kind and a numerical method is developed on the basis of the regularized equation. The convergence of the numerical solution to the exact solution of the Volterr integral equation of the first kind is proved, the estimates of the error and the recursive formula of the computational process are obtained.

In questions of interpolation of functions of many variables, the theoretic – numerical grids make it possible to obtain interpolation formulas, the accuracy of which increases with increasing smoothness of functions, and the number of variables does not significantly affect the order of smallness of the remainder term. Applying quadrature formulas constructed using parallelepipedal grids to the Fourier coefficients of the function $f \in E_s^\alpha$, we obtain the interpolation formula

$$f(x_1,...,x_s) = \frac{1}{N}\sum_{k=1}^{N} f\left(\frac{ka_1}{N},...,\frac{ka_s}{N}\right)\psi_k(x_1,...,x_s) + O\left(N^{-\frac{\alpha-1}{2}} \ln^{\frac{\alpha+1}{2}s-1} N\right),$$

where functions $\psi_k(x_1,...,x_s)$ are defined by equality

$$\psi_k(x_1,...,x_s) = \sum_{\overline{m}_1,...,\overline{m}_s < \sqrt{N}\ln^{-\frac{s}{2}} N} e^{2\pi i\left[m_1\left(x_1 - \frac{ka_1}{N}\right)+...+m_s\left(x_s - \frac{ka_s}{N}\right)\right]}.$$

This formula is exact for trigonometric polynomials, the degree of which does not exceed the value $\sqrt{N}\ln^{-\frac{s}{2}} N$.

More accurate results in the interpolation of functions of many variables can be obtained in another way, based on the representation of a function by some finite sum of integrals and then applying the corresponding quadrature formulas to these integrals.

4. Conclusion

The application of theoretic – numerical methods to the problems of approximate analysis is reduced to the use of non-uniform and parallelepipedal grids. Theoretic – numerical grids make it possible to obtain interpolation formulas, the accuracy of which increases with increasing smoothness of functions. The number of variables does not significantly affect the order of smallness of the residual term.

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ЕРКІН ЕСЕЛІ ИНТЕГРАЛДАРДЫ ЕСЕПТЕУ ҮШІН КВАДРАТУРАЛЫҚ ФОРМУЛАЛАРДЫ ҚОЛДАНУ ТУРАЛЫ

Аннотация. Бұл мақалада біркелкі емес торлар, Монте-Карло және оңтайлы коэффициенттер әдісте-рімен еркін еселі интегралдарды есептеу қарастырылды. Көп өлшемді интегралдарды есептеудің көрсетілген сандық әдістеріне салыстырмалы талдау жасалды. Оңтайлы коэффициенттер әдісі басқа әдістермен салыстырғанда артықшылыққа ие екендігі анықталды. Шамамен талдау мәселелеріне теориялық-сандық әдістерді қолдану саласында біркелкі емес және параллелепипедалды торларды пайдалану нәтижелердің көпшілігінің негізі болып табылатыны көрсетілген. Көп айнымалылы функцияларды теориялық-сандық торлармен интерполяциялау функциялардың тегістігін арттырумен өсетін интерполяциялық формулаларды алуға мүмкіндік беретіні анықталды. Бұл жағдайда айнымалылардың саны қалдық мүшенің аздығы тәртібіне елеулі әсер етпейді. Фурье коэффициенттеріне $f \in E_s^\alpha$ функцияны пайдалану параллелепипедалды торлар арқылы құрылатын квадратуралық формулалардан интерполяциялық формуланы алуға мүмкіндік береді. Мұндай формула дәрежесі $\sqrt{N}\ln^{-\frac{s}{2}} N$ мәннен аспайтын тригонометриялық полиномдар үшін дәл болып табылады.

Түйін сөздер: теориялық-сандық әдісі, квадратуралық формула, оңтайлы коэффициенттер әдісі, көп өлшемді интегралдар.

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О ПРИМЕНЕНИИ КВАДРАТУРНЫХ ФОРМУЛ ДЛЯ ВЫЧИСЛЕНИЯ ИНТЕГРАЛОВ ПРОИЗВОЛЬНОЙ КРАТНОСТИ

Аннотация. В данной работе рассмотрено вычисление интегралов произвольной кратности методами: неравномерных сеток, Монте-Карло и оптимальных коэффициентов. Был сделан сравнительный анализ указанных численных методов интегрирования многократных интегралов. Установлено, что метод оптимальных коэффициентов обладает преимуществом по сравнению с остальными методами. Показано, что использование неравномерных и параллелепипедальных сеток составляет основу большинства результатов, полученных в области применения теоретико-числовых методов к вопросам приближенного анализа. Установлено, что интерполяция функций многих переменных теоретико-числовыми сетками позволяет получить интерполяционные формулы, точность которых возрастает с увеличением гладкости функций. Число переменных в этом случае не оказывает существенного влияния на порядок малости остаточного члена. Использование функции $f \in E_s^{\alpha}$ к коэффициентам Фурье позволяет получить интерполяционную формулу из квадратурных формул, которые построены с помощью параллелепипедальных сеток. Такая формула точна для тригонометрических полиномов, степень которых не превосходит величины $\sqrt{N} \ln^{-\frac{s}{2}} N$.

Ключевые слова: теоретико-числовой метод, квадратурная формула, метод оптимальных коэффициентов, многократные интегралы.

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REFERENCES

[1] Korobov N. M. (1963) Teoretiko-chislovye metody v priblizhennom analize. *M.: Fizmat-Giz*. (in Russ.)

[2] Dobrovol'skij N., Korobov N. M. (**2002**) *Chebyshev collection* [*Chebyshevskij sbornik*]. 3:1, 41-48. (in Russ.)

[3] Korobov I. M. (**2002**) Ob odnoj ocenke v metode optimal'nyh kojefficientov. *Tezisy IV Vserossijskoj konferencii "Sovremennye problemy matematiki, mehaniki, informatiki"*, Tula, Rossiia. P. 39-40 (in Russ.)

[4] Korobov N. M. (**1957**) *DAN SSSR* 115:6, 1062-1065 (in Russ.)

[5] Korobov N. M. (**1959**) *DAN SSSR* 124:6, 1207-1210 (in Russ.)

[6] Voronin S. M. (**1994**) *Russian Academy of Sciences. Izvestiya: Mathematics* 58:5,189-194 (in Russ.); *Russian Academy of Sciences. Izvestiya Mathematics,* 45:2,417-422 http://dx.doi.org/10.1070/IM1995v045n02ABEH001657

[7] Voronin S. M. (**1995**) *Izvestiya RAN. Serija matematicheskaja* 59:4, 3–8 (in Russ.); *Izvestiya: Mathematics* 59:4, 665-670 http://dx.doi.org/10.1070/IM1995v059n04ABEH000028

[8] S.K. Zamanova, A.I. Kupchishin, A.D. Muradov (**2008**) *News of the National Academy of Sciences of the Republic of Kazakhstan Series of physico-mathematical sciences* [*Izvestija Nacional'noj akademii nauk RK Serija fiziko-matematicheskaja*] 6:262, 22-26. (in Russ.)

[9] S.K. Zamanova, A.I. Kupchishin, A.D. Muradov (**2009**) *News of the National Academy of Sciences of the Republic of Kazakhstan Series of physico-mathematical sciences* [*Izvestija Nacional'noj akademii nauk RK Serija fiziko-matematicheskaja*] 5:267, 38-41 (in Russ.)

[10] S.K. Zamanova, A.I. Kupchishin, A.D. Muradov (**2010**) *News of the National Academy of Sciences of the Republic of Kazakhstan Series of physico-mathematical sciences* [*Izvestija Nacional'noj akademii nauk RK Serija fiziko-matematicheskaja*] 3:271, 73-79 (in Russ.)

[11] Zamanova S.K. (**2012**) CALCULATION TENFOLD INTEGRAL by METHOD of OPTIMAL COEFFICIENTS with PROVISION for NUMBERS of KNOTS QUADRATURE FORMULAS. *Proceedings ICKIICE 2012, Materials of the International Conference of Korea Institute of Information & Communication Engineering, June 28 - 29, Istanbul, Turkey.* 5:1. P. 62-66

[12] Zamanova S.K. (**2013**) Comparative characteristics of calculated sets of optimal coefficients and values fifteen fold integrals by number theoretic methods with provision for numbers of knots quadrature formulas. *Proceedings the IV International Conference on European Science and Technology, Munich, Germany.* P. 728-734.

[13] Zamanova S.K. (**2013**) Calculation fifteenfold integrals by method of optimal coefficients for small values of the numbers of knots quadrature formulas. *Proceedings ICFICE 2013*, *Future Information Communication Technology and Applications*, *Springer, Netherlands, XXV.* I:V. P. 453-460

[14] Zamanova S.K. (**2014**) About the calculatoin of fifteenfold integrals. *The V Congress of the Turkic World Mathematicians, June 5-7, Bishkek, Kyrgyzstan.* P. 82

[15] Demidov S.S., Morozova E.A., Chubartkov V.N. i dr. (**2017**) *Chebyshev collection* [*Chebyshevskij sbornik*] 18:4, 6-85 (in Russ.) https://doi.org/10.22405/2226-8383-2017-18-4-6-85

[16] T.T. Karakeev, N.T. Mustafayeva (**2018**) *News of the National Academy of Sciences of the Republic of Kazakhstan Series of physico-mathematical sciences* 5:321, 10-18 (in Eng.). ISSN 2518-1726 (Online), ISSN 1991-346X (Print) https://doi.org/10.32014/2018.2518-1726.2

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SOLITON SOLUTIONS FOR THE (2+1)-DIMENSIONAL INTEGRABLE FOKAS-LENELLS EQUATION

Abstact. Studying of solitons led to the discovery of a number of new directions related to it. There is interest in which is also enhanced in connection with the discovery of new examples in which soliton processes are manifested. The number and variety of nonlinear equations containing solitons as the most interesting solutions significantly increase due to generalizations to the two-dimensional and three-dimensional cases. Such popular transformations as Darboux, Backlund and Hirota's bilinear method are often used to find exact different kind of the solutions of nonlinear equations.

In the present paper, we present Lax pair of the (2+1)-dimensional integrable Fokas-Lenells equation. The bilinear form of the (2+1)-dimensional integrable Fokas-Lenells equation was obtained by the Hirota's bilinear method. By using Hirota's bilinear method, we construct exact one-soliton and two-soliton solutions of the (2+1)-dimensional Fokas-Lenells equation. The graphics of the obtained solutions are presented. The obtained new results have important physical applications.

Keywords: Hirota method, Lax representation, soliton solution, Fokas-Lenells equation.

Introduction. In many areas of science, the object of intensive theoretical and experimental research is soliton which is mean the "solitary" wave (solitary wave). Soliton can be used to transmit information, where the main idea is to use in each bit interval to represent units in the stream of binary signals. Mathematically, solitons are localized stationary solutions of nonlinear partial differential equations or their generalizations.

One of the generalization of nonlinear Schrödinger is the integrable (1+1)-dimensional Fokas-Lenells equation (FL) [1, 2] which was proposed by J. Fokas and A.S. Lenells. At present, studying multidimensional integrable systems containing derivatives in more than two variables is very interesting and relevant and namely finding exact soliton solutions. In this regard, by considering the (1+1)-dimensional FL equation, we obtain the (2+1)-dimensional FL equation and present its Lax pair [1, 2].

The paper is organized as follows. In Section 2, we present Lax representation of the (2+1)-dimensional FL equation. Hirota method is presented in Section 3. Namely, we apply Hirota method for FL equation and find one soliton and two soliton solutions by obtained bilinear equation. In Section 4, we give conclusion.

Lax representation. The (2+1)-dimensional FL equation is given by next form

$$iq_{xt} - iq_{xy} + 2q_x - q_x|q|^2 + iq = 0, \tag{1}$$

where q is the complex shell of the field, the indices *x, y* and *t* denote the partial derivatives with respect to the arguments *x, y* and *t*, and i is the complex number.

To construct solutions of differential equations, a number of conditions must be fulfilled, one of which is to satisfy the compatibility condition [1]. The studied equation (1) satisfies the "compatibility condition" and has the following has a Lax representation

$$\Psi_x = U\Psi, \qquad U = -i\lambda^2\sigma_3 + \lambda Q,$$

$$\Psi_t = \Psi_y + W\Psi, \qquad W = W_0 + \frac{1}{\lambda}W_{-1} - \frac{i}{4\lambda^2}\sigma_3,$$

where Ψ = Ψ (x,t) is a 2×2 matrix-valued eigenfunction, λ is an isospectral parameter, and matrices are given in the form:

$$Q = \begin{pmatrix} 0 & q_x \\ r_x & 0 \end{pmatrix}, \quad W_0 = i\sigma_3 - \frac{iqr}{2}\sigma_3, \quad W_{-1} = \frac{i}{2}\begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hirota's bilinear method. Soliton solutions of the (2+1)-dimensional FL equation will be constructed by the so-called bilinear Hirota's method. The outline for constructing soliton solutions by this method is as follows:

1. The first we apply the dependent variable transformation for FL equation in order to obtain its bilinear form.
2. The second we consider the formal series of perturbation theory.
3. The third we build multi soliton solutions.

One-soliton solution of the (2+1)-dimensional FL equation. In order to construct soliton solutions of equation (1), we use the bilinear form of FL equation has the form [6] $q = \frac{g}{f}$. Then the bilinear form of the FL equation

$$iD_xD_t(g\cdot f) - iD_xD_y(g\cdot f) + 2D_x(g\cdot f) + igf = 0, \tag{2a}$$

$$iD_xD_t(f\cdot f^*) - iD_xD_y(f\cdot f^*) - \frac{1}{2}D_x(g\cdot g^*) = 0, \tag{2б}$$

$$iD_t(f\cdot f^*) - iD_y(f\cdot f^*) - \frac{1}{2}g\cdot g^* = 0, \tag{2в}$$

$$iD_t(f\cdot f_x^*) - iD_y(f\cdot f_x^*) - \frac{1}{2}g\cdot g_x^* = 0, \tag{2г}$$

where g is complex function, f is real one, the "*" sign means complex conjugation and D_x, D_y, D_t, are bilinear differential operators which defined by

$$D_x^m D_t^n f\cdot f^* = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x,t)f^*(x',t')\,|_{x'=x,t'=t},$$

where x', t' as two formal variables, $f(x,t)$ and $f^*(x',t')$ - two functions, m and n - two non-negative integers.

Then, we expand the functions g and f in bilinear equation (2) with respect to small parameter ε as follows [2]:

$$g(x,y,t) = \varepsilon g_1(x,y,t) + \varepsilon^3 g_3(x,y,t) + \ldots, \tag{3a}$$

$$f(x,y,t) = 1 + \varepsilon^2 f_2(x,y,t) + \varepsilon^4 f_4(x,y,t) + \ldots, \tag{3б}$$

where g_j is complex function, f_n is real ones (j = 1,3,5,...; n = 2,4,6,...).

In the case of a one-soliton solution of equation (1), the formal parameter in equation (3) is taken as $\varepsilon = 1$ and $j = 1,\ n = 2$, i.e.

$$g = \varepsilon g_1, \tag{4a}$$

$$f = 1 + \varepsilon^2 f_2 \tag{4б}$$

and we find the solution according to the following statement:

$$q = \frac{g}{f} = \frac{g_1}{1 + f_2}, \tag{5}$$

where $g_1 = e^{\theta_1}$, $\theta_1 = a_1 x + b_1 t + c_1 y + d_1$ and $\theta_1^* = a_1^* x + b_1^* t + c_1^* y + d_1^*$, here a_1, b_1, c_1, d_1 - complex constants.

Substituting expression (4) into equation (2) and collecting in powers of the parameter ε, we get next system

$$\varepsilon^1 \ :\ (iD_x D_t - iD_x D_y + 2D_x)(g_1 \cdot 1) + ig_1 = 0, \tag{6}$$

$$\varepsilon^3 \ :\ (iD_x D_t - iD_x D_y + 2D_x)(g_1 \cdot f_2) + ig_1 f_2 = 0, \tag{7}$$

$$\varepsilon^2 \ :\ i(D_x D_t - D_x D_y)(f_2 + f_2^*) - \frac{1}{2} D_x (g_1 \cdot g_1^*) = 0, \tag{8}$$

$$\varepsilon^4 \ :\ iD_x D_t (f_2 \cdot f_2^*) - iD_x D_y (f_2 \cdot f_2^*) = 0, \tag{9}$$

$$\varepsilon^2 \ :\ i(D_t - D_y)(f_2 - f_2^*) - \frac{1}{2} g_1 \cdot g_1^* = 0, \tag{10}$$

$$\varepsilon^4 \ :\ i(D_t - D_y)(f_2 \cdot f_2^*) = 0, \tag{11}$$

$$\varepsilon^2 \ :\ (-iD_x D_t + iD_x D_y)(f_2^* \cdot 1) - \frac{1}{2} g_1 g_{1x}^* = 0, \tag{12}$$

$$\varepsilon^4 \ :\ i(D_x - D_y)(f_2 \cdot f_{2x}^*) = 0. \tag{13}$$

Now, applying the properties of the Hirota operator to equations (6) - (13), we obtain

$$\varepsilon^1 \ :\ ig_{1tx} - ig_{1yx} + 2g_{1x} + ig_1 = 0, \tag{14}$$

$$\begin{aligned} \varepsilon^3 \ :\ & ig_{1tx} f_2 - ig_{1t} f_{2x} - ig_{1x} f_{2t} + ig_1 f_{2tx} - ig_{1yx} f_2 + +ig_{1y} f_{2x} + \\ & + ig_{1x} f_{2y} - ig_1 f_{2yx} + 2g_{1x} f_2 - 2g_1 f_{2x} + ig_1 f_2 = 0 \end{aligned} \tag{15}$$

$$\varepsilon^2 \ :\ if_{2tx} + if_{2tx}^* - if_{2yx} - if_{2yx}^* - \frac{1}{2} g_{1x} g_1^* + \frac{1}{2} g_1 g_{1x}^* = 0, \tag{16}$$

$$\begin{aligned} \varepsilon^4 \ :\ & if_{2tx} f_2^* - if_{2t} f_{2x}^* - if_{2x} f_{2t}^* + if_2 f_{2tx}^* - if_{2yx} f_2^* + \\ & + if_{2y} f_{2x}^* + if_{2x} f_{2y}^* - if_2 f_{2yx}^* = 0, \end{aligned} \tag{17}$$

$$\varepsilon^2 \quad : \quad if_{2t} - if_{2t}^* - if_{2y} + if_{2y}^* - \frac{1}{2} g_1 g_1^* = 0, \tag{18}$$

$$\varepsilon^4 \quad : \quad if_{2t} f_2^* - if_2 f_{2t}^* - if_{2y} f_2^* + if_2 f_{2y}^* = 0, \tag{19}$$

$$\varepsilon^2 \quad : \quad -if_{2xt}^* + if_{2xy}^* - \frac{1}{2} g_1 g_{1x}^* = 0, \tag{20}$$

$$\varepsilon^4 \quad : \quad if_{2t} f_{2x}^* - if_2 f_{2xt}^* - if_{2y} f_{2x}^* + if_2 f_{2xy}^* = 0. \tag{21}$$

By solving the system of equation (14)-(21) we can get from equation (14)

$$a_1 = \frac{1}{c_1 - b_1 + 2i}, \tag{22}$$

the equation (15) gives conjugation form of a_1

$$a_1^* = \frac{1}{c_1^* - b_1^* - 2i}, \tag{23}$$

and from equations (16) - (21) we obtain

$$f_2 = \frac{ia_1^2 a_1^*}{2(a_1 + a_1^*)^2} e^{\theta_1 + \theta_1^*}. \tag{24}$$

Then by substituting equation (24) into equation (5), we obtain the one-soliton solution of the (2+1)-dimensional FL equation, which is

$$q = \frac{e^{\theta_1}}{1 + \frac{ia_1^2 a_1^*}{2(a_1 + a_1^*)^2} e^{\theta_1 + \theta_1^*}} = \frac{2(a_1 + a_1^*)^2}{2(a_1 + a_1^*)^2 e^{-\theta_1} + ia_1^2 a_1^* e^{\theta_1^*}}, \tag{25}$$

or if rewrite equation (25), with $\theta_1 = \kappa_1 + \chi_1$, then we get

$$q = \frac{4\alpha_1^2 e^{i\chi_1}}{4\alpha_1^2 e^{-\kappa_1} + \frac{1}{2}(\alpha_1^2 + \beta_1^2)(i\alpha_1 - \beta_1) e^{\kappa_1}}, \tag{26}$$

where

$$\chi_1 = \beta_1 x + \nu_1 t + \tau_1 y + n_1, \quad \kappa_1 = \alpha_1 x + \mu_1 t + \sigma_1 y + m_1$$

with

$$a_1 = \alpha_1 + i\beta_1, \ \ b_1 = \mu_1 + i\nu_1, \ \ c_1 = \sigma_1 + i\tau_1, \ \ d_1 = m_1 + in_1.$$

Finally, the equation (26) has the form

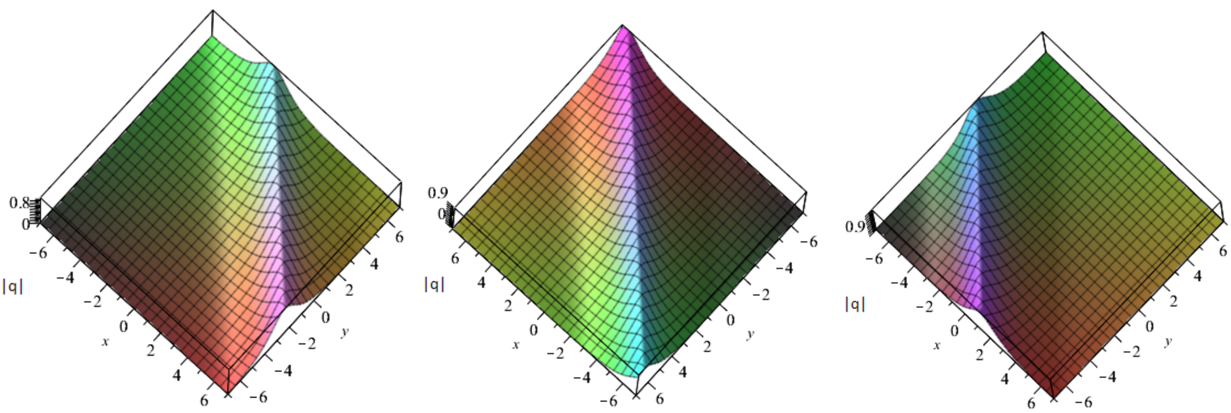
$$q=\frac{p^2 e^{i\chi_1}}{p\sqrt{h}\left(\frac{\sqrt{h}}{p}e^{\kappa_1}+\frac{p}{\sqrt{h}}e^{-\kappa_1}\right)}, \tag{27}$$

where $p=2\alpha_1$ and $h=\frac{i}{2}(\alpha_1^2+\beta_1^2)\alpha_1-\frac{1}{2}(\alpha_1^2+\beta_1^2)\beta_1$.

In addition, when $\frac{\sqrt{h}}{p}=e_1^{\delta}$ and $\frac{p}{\sqrt{h}}=e^{-\delta_1}$ the equation (27) can be rewritten as

$$q=\frac{pe^{i\chi_1}}{2\sqrt{h}}sech(\kappa_1+\delta_1)=\frac{pe^{i\chi_1}}{2\sqrt{h}\cosh(\kappa_1+\delta_1)}. \tag{28}$$

So, the equation (28) is one-soliton solution to the (2+1)-dimensional FL equation. Plot of the one-soliton solution is presented in Figure 1.



(a) (b) (c)

Figure 1 - Dynamics of the one-soliton solution with next parameters:
$a_1=1-i$, $c_1=1-i$, $b_1=1-i$ and $d_1=1-i$. $t=-5$ (a); t = 0 (b); t = 5 (c).

Two-soliton solution of the (2+1)-dimensional FL equation. To find the two-soliton solution of equation (1), the formal parameter in equation (3) is taken, as $\varepsilon=1$ and $j=1,3,\ n=2,4$, i.e.

$$\varepsilon^1 \quad : \quad (iD_xD_t-iD_xD_y+2D_x)(g_1\cdot 1)+ig_1=0, \tag{30}$$

$$\varepsilon^3 \quad : \quad (iD_xD_t-iD_xD_y+2D_x)(g_1\cdot f_2+g_3\cdot 1)+ig_1f_2+ig_3=0, \tag{31}$$

$$\varepsilon^5 \quad : \quad (iD_xD_t-iD_xD_y+2D_x)(g_1\cdot f_4+g_3\cdot f_2)+ig_1f_4+ig_3f_2=0, \tag{32}$$

$$\varepsilon^7 \quad : \quad (iD_xD_t-iD_xD_y+2D_x)(g_3\cdot f_4)+ig_3f_4=0, \tag{33}$$

$$\varepsilon^2 \quad : \quad i(D_xD_t-D_xD_y)(f_2+f_2^*)-\frac{1}{2}D_x(g_1\cdot g_1^*)=0, \tag{34}$$

$$\varepsilon^4 \quad : \quad iD_xD_t(f_2\cdot f_2^*+f_4+f_4^*)-iD_xD_y(f_2\cdot f_2^*+f_4+f_4^*)-$$

$$-\frac{1}{2}D_x(g_1g_3^* + g_3g_1^*) = 0, \quad (35)$$

$$\varepsilon^6 \; : \; iD_xD_t(f_2 \cdot f_4^* + f_4 \cdot f_2^*) - iD_xD_y(f_2 \cdot f_4^* + f_4 \cdot f_2^*) -$$

$$-\frac{1}{2}D_x(g_3g_3^*) = 0, \quad (36)$$

$$\varepsilon^8 \; : \; iD_xD_t(f_4 \cdot f_4^*) - iD_xD_y(f_4 \cdot f_4^*) = 0, \quad (37)$$

$$\varepsilon^2 \; : \; i(D_t - D_y)(f_2 - f_2^*) - \frac{1}{2}g_1 \cdot g_1^* = 0, \quad (38)$$

$$\varepsilon^4 \; : \; i(D_t - D_y)(f_2 \cdot f_2^*) + i(D_t - D_y)(f_4 - f_4^*) - \frac{1}{2}(g_1 \cdot g_3^* + g_1^*g_3) = 0, \quad (39)$$

$$\varepsilon^6 \; : \; i(D_t - D_y)(f_2 \cdot f_4^*) - i(D_t - D_y)(f_2^* \cdot f_4) - \frac{1}{2}(g_3 \cdot g_3^*) = 0, \quad (40)$$

$$\varepsilon^8 \; : \; i(D_t - D_y)(f_4 \cdot f_4^*) = 0. \quad (41)$$

$$\varepsilon^2 \; : \; (-iD_xD_t + iD_xD_y)(f_2^* \cdot 1) - \frac{1}{2}g_1g_{1x}^* = 0, \quad (42)$$

$$\varepsilon^4 \; : \; i(D_x - D_y)(f_2 \cdot f_{2x}^*) - i(f_{4xt}^* - f_{4xy}^*) - \frac{1}{2}(g_1g_{3x}^* + g_3g_{1x}^*) = 0, \quad (43)$$

$$\varepsilon^6 \; : \; i(D_t - D_y)(f_2 \cdot f_{4x}^* + f_4 \cdot f_{2x}^*) - \frac{1}{2}g_3g_{3x}^* = 0, \quad (44)$$

$$\varepsilon^8 \; : \; i(D_t - D_y)(f_4 \cdot f_{4x}^*) = 0. \quad (45)$$

By applying the properties of the Hirota operator we solve the system of equation (30)-(45), and can get the two-soliton solution of equation (1), which has the next form

$$q = \frac{g_1 + g_3}{1 + f_2 + f_4}, \quad (46)$$

where

$$g_1 = e^{\theta_1} + e^{\theta_2},$$
$$g_3 = k_1 e^{\theta_1 + \theta_2 + \theta_1^*} + k_2 e^{\theta_1 + \theta_2 + \theta_2^*},$$
$$f_2 = l_1 e^{\theta_1 + \theta_1^*} + l_2 e^{\theta_1 + \theta_2^*} + l_3 e^{\theta_1^* + \theta_2} + l_4 e^{\theta_2 + \theta_2^*},$$
$$f_4 = m e^{\theta_1 + \theta_1^* + \theta_2 + \theta_2^*},$$

with

$$\theta_1 = a_1 x + b_1 t + c_1 y + d_1, \qquad \theta_1^* = a_1^* x + b_1^* t + c_1^* y + d_1^*,$$

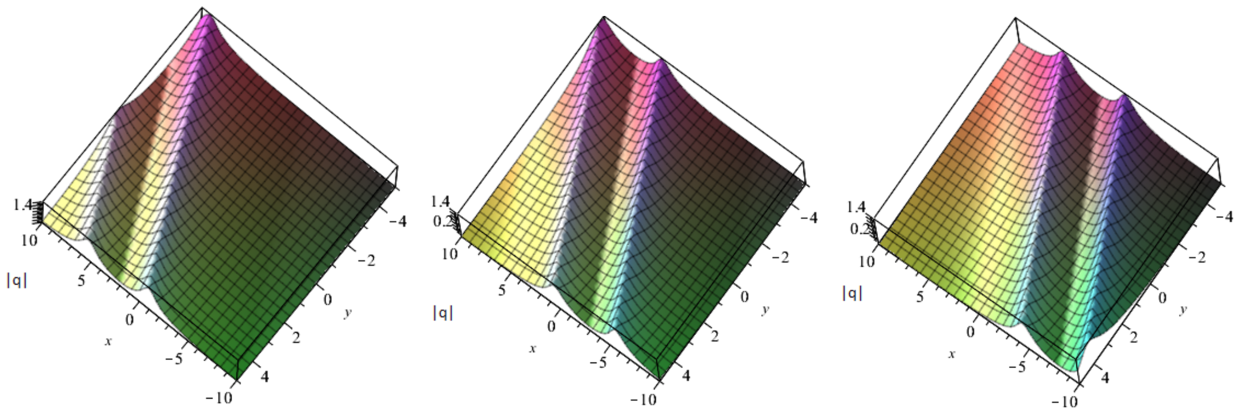
$$\theta_2 = a_2 x + b_2 t + c_2 y + d_2, \quad \theta_2^* = a_2^* x + b_2^* t + c_2^* y + d_2^*,$$

$$l_1 = \frac{i a_1^2 a_1^*}{2(a_1 + a_1^*)^2}, \quad l_2 = \frac{i a_1^2 a_2^*}{2(a_1 + a_2^*)^2}, \quad l_3 = \frac{i a_2^2 a_1^*}{2(a_1^* + a_2)^2}, \quad l_4 = \frac{i a_2^2 a_2^*}{2(a_2 + a_2^*)^2},$$

$$k_1 = \frac{i(a_1^*)^3 (a_2 - a_1)^2}{2(a_1^* + a_2)^2 (a_1 + a_1^*)^2}, \quad k_2 = \frac{i(a_2^*)^3 (a_2 - a_1)^2}{2(a_1 + a_2^*)^2 (a_2 + a_2^*)^2},$$

$$m = -\frac{(a_2 - a_1)^2 (a_2^* - a_1^*)^2 (a_1 + a_2 + a_1^* + a_2^*) a_1^* a_2^* a_1^2 a_2^2}{2(a_1 + a_1^*)^2 (a_2 + a_2^*)^2 (a_1 + a_2^*)^2 (a_2 + a_1^*)^2},$$

and a_n, b_n, c_n, d_n - complex constants, $n = 1,2$.



(a) (b) (c)

Figure 2 - Dynamics two-soliton solution with next parameters:
$a_1 = 0.5 + 0.5i$, $c_1 = 0.5 + 0.5i$, $b_1 = 0.5 + 0.5i$ and $d_1 = 0.5 + 0.5i$. $t = -5$ (a); t = 0 (б); t = 5 (c).

Conclusion. Thus, we studied (2+1)-dimensional Fokas-Lenells equation by the Hirota's bilinear method which is considered one of the effective methods for finding exact solutions of integrable equations. By using this method, exact one-soliton and two-soliton solutions of the (2+1)-dimensional FL equation are constructed. Additionally, we present the graphical representation of the obtained soliton solutions.

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(2+1)-ӨЛШЕМДІ ИНТЕГРАЛДАНАТЫН ФОКАС-ЛЕНЭЛЛС ТЕҢДЕУІНІҢ СОЛИТОНДЫ ШЕШІМДЕРІ

Аннотация. Солитондарды зерттеу солитонмен байланысты бірқатар жаңа бағыттардың ашылуына алып келді. Сонымен қатар, солитонды процестер байқалатын жаңа бағыттардың ашылуымен байланысты қызығушылықтар бар. Солитон құрайтын сызықты емес теңдеулердің саны мен алуандығы екі қлшемді және үш өлшемді жалпылауларға көшу арқылы күннен-күнге артуда. Дарбу, Бэклунд түрлендіруі және Хиротаның бисызықты әдісі сияқты танымал түрлендірулер сызықты емес теңдеулердің әр түрлі шешімдерін табу кезінде жиі қолданылады.

Демек, берілген жұмыс алдыңғы табылған (2+1)-өлшемді интегралданатын Фокас-Ленэлсс теңдеуі және оның бисызықты түрі атты жұмысымыздың жалғасы болып табылады. Енді, зерттеулерімізді жалғастыра отырып, Хирота әдісі арқылы интегралданатын (2+1)-өлшемді Фокас-Ленэлсс теңдеуінің 1-солитонды және 2-солитонды шешімдері табылып, графиктері тұрғызылды. Біздің тапқан нәтижелеріміз маңызды физикалық

қолданысқа ие.

Түйін сөздер: Хирота әдісі, Лакс көрінісі, солитонды шешім, Фокас-Ленэллс теңдеуі.

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СОЛИТОННЫЕ РЕШЕНИЯ (2+1)-МЕРНОГО ИНТЕГРИРУЕМОГО УРАВНЕНИЯ ФОКАСА-ЛЕНЭЛЛСА

Аннотация. Успехи в исследовании солитонов привели к открытию целого ряда новых направлений связанных с ним, тем вдохновили бурной активностью исследователей в данных направлениях. Кроме того, интерес, к которому усиливается также в связи обнаружениями новых примеров, в которых проявляются солитонные процессы. Количество и разнообразие нелинейных уравнений, содержащих солитоны в качестве наиболее интересных решений, существенно увеличиваются благодаря обобщениям на двумерные и трехмерные случаи. Для нахождения солитонных решений нелинейных уравнений часто применяются такие популярные преобразования, как Дарбу, Бэклунда и метод Хироты.

Таким образом, данная работа является продолжением нашей предыдущей работы, в которой было найдено (2+1)-мерное интегрируемое уравнение Фокаса-Ленэллса и построена ее билинейная форма методом Хироты. Теперь, продолжая наши исследования, методом Хироты найдены его точные 1-солитонное и 2-солитонное решения с помощью уже полученной нами билинейной формы (2+1)-мерного уравнения Фокаса-Ленэллса и построены их графики. Найденные нами результаты имеют важные физические приложения.

Ключевые слова: метод Хироты, представление Лакса, солитонное решение, уравнение Фокаса-Ленэллса.

REFERENCES

[1] Ablowitz M., Sigur H. Solitons and method of the inverse problem [Solitony I metod obratnoj zadachi]. M., 1987-479 p.

[2] Zakharov V.E., Manakov S.V., Novikov S.P., Pitaevsky L.P. Introduction to the theory of integrable systems [Vvedenie v teoriyu integriruemyh sistem]. M: Nauka, 1980. -320 p. (In Russian).

[3] Jingsong H., Shuwei X., Kuppuswamy P. Rogue Waves of the Fokas-Lenells Equation. Journal of the Physical Society of Japan. **81**, 124007 **(2012)**.

[4] Zhassybayeva M.B., Nugmanova G.N. Spin system equivalent to the integrable Fokas-Lenells equation [Spinovaya sistema, ehkvivalentnaya integriruemomu uravneniyu Fokasa-Lenehllsa] Bulletin of L.N. Gumilyov Eurasian National University [Vestnik Evrazijskogo nacional’nogo universiteta imeni L.N. Gumileva]. 1(122), 53-57, 2018.

[5] Tasmambetov Zh.N., Rajabov N., Ubayeva Zh.K. (2019) The coordinated solution of two differential equations in private derivatives of the third order \\ Bulletin of national academy of sciences of the Republic of Kazakhstan. Volume 4, Number 326 (2019), PP.83-91. https://doi.org/10.32014/2019.2518-1726.47 ISSN 2518-1726 (Online), ISSN 1991-346X (Print).

[6] Zhassybayeva M., Yesmakhanova K. The construction of the (2+1)-dimensional integrable Fokas-Lenells equation and its bilinear form by Hirota method. International Conference on Technology, Engineering and Science. October 26-29, 2018, Antalya, Turkey [in print].

[7] Matsuno Y.A. direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation: I. Bright soliton solutions. Journal of Physics A: Mathematical and Theoretical. 45, 23202 (2012).

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SOME QUESTIONS ON EXTERNAL DEFINABILITY

Abstract. The article discusses the various approaches to the concept of external definability developed in o-minimal theories. An example of o-minimal theories shows how external constants help determine the existence of a solution in a model of a formula with external constants. The basic concepts are formulated with the help of which external definability is proved. A brief review of the results for dependent theories is given. In conclusion, sufficient conditions are formulated so that the NSOP theory has the some property of external definability. A brief explanation of the stated theorem is given.
Keywords: externally definable, neighborhood of tuple of the set in the type, non orthogonality of two types.

External definability. Let $\mathfrak{M}$ be elementary substructure of $\mathfrak{N}$. Let $\bar{\alpha} \in N\backslash M$ and $p := tp(\bar{\alpha}|M)$. Then for any formula $\psi(\bar{x}, \bar{y})$ define the predicate $R_{(\psi,p)}(\bar{y})$ on the set M, $\vDash R_{(\psi,p)}(\bar{a})$ iff $\psi(\bar{x}, \bar{a}) \in tp(\bar{\alpha}|M)$ iff $\mathfrak{N} \vDash \psi(\bar{\alpha}, \bar{a})$. Denote by $\mathfrak{M}^+ = \langle M; \Sigma^+\rangle$, where $\Sigma^+ := \{R_{(\psi,p)}(\bar{y}) | p \in S(M),\ \psi \in \Sigma\}$. It follows from definition that if a pair of models (M, N) is conservative pair (type of any tuple elements from N over M is definable), then the structure $\mathfrak{M}^+$ is the structure obtained from $\mathfrak{M}$ scolemisation of $\mathfrak{M}$. We will consider the simple cases when $\mathfrak{M}^+$ constructed from one 1-type for o-minimal theory from two approaches.

Let $\mathfrak{M}$ be a model of an arbitrary complete theory T of the signature Σ. We say that $\mathfrak{M}_p^+$ is expansion of $\mathfrak{M}$ by type $p \in S_1(M)$, if $\mathfrak{M}_p^+ := \langle M; \Sigma_p^+\rangle$, where $\Sigma_p^+ := \{R_{(\psi,p)}(\bar{y}) | \psi \in \Sigma\}$.

We say that $\mathfrak{M}_p^+$ *admits uniformly representation of* Σ_p^+-formulas by Σ-formulas, if for any formula $\phi(\bar{y})$ of Σ_p^+ there exists Σ-formula $K_\phi(\bar{y}, \bar{z})$, there exists $\bar{\alpha} \in N\backslash M$ such that for any $\bar{a} \in M$ the following holds:

$$\mathfrak{M}_p^+ \vDash \phi(\bar{a}) \Leftrightarrow \vDash K_\phi(\bar{a}, \bar{\alpha}).$$

Approach of Macpherson-Marker-Steinhorn. In the paper [1] (preprint 1994 Macpherson-Marker-Steinhorn proved weak o-minimality of the expansion of an o-minimal structure by unary convex predicate, such that the predicate is traversed by a uniquely realizable 1-type. Following D. Marker [5], an uniquely realizable 1-type $p \in S_1(M)$ over model is that prime model over model and one realization of this 1-type p contains just this element from the set of realization of the type. An uniquely realizable 1-type has the next property: there is no definable function acting on the set of realizations of this 1-type p. Macpherson-Marker-Steinhorn considered at the same time two structures $\mathfrak{M}^+ = \langle M; \Sigma \cup \{U^1\}\rangle$ and $\mathfrak{N} = \langle N; \Sigma\rangle$, where $\mathfrak{N}$ is a model of an o-minimal theory of the signature Σ and a saturated elementary extension of $\mathfrak{M}$. They defined a new unary convex predicate U by using an element $\alpha \in N\backslash M$ from the set of realizations of an irrational $1-$type $p \in S_1(M)$ such that for every $a \in M$ the following holds:

$$\mathfrak{M}^+ \vDash U(a) \Leftrightarrow \mathfrak{N} \vDash a < \alpha.$$

By induction of construction of formulas $\phi(\bar{y})$ of the signature $\Sigma^+ = \Sigma \cup \{U^1\}$ there is a formula $K_\phi(\bar{y}, \alpha)$ of the signature Σ such that for any $\bar{a} \in M$ the following holds:

for any $n, k, m < \omega$,

$$QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) < QV_{p_n}(\alpha_{n+k+1}, \alpha_{n+k+2}, \dots, \alpha_{n+k+m}),$$

and consequently, all these sets have empty intersection. D. Marker proved that for any set A in o-minimal theory, for any $q, r \in S_1(A)$ if q and r are not weakly orthogonal than there is $A-$definable monotonic bijection from $q(\mathfrak{N})$ to $r(\mathfrak{N})$. Then for $r \in S_1(M)$, p_i non weakly orthogonal to r, $QV_r(\bar{\alpha}) \cap QV_r(\bar{\alpha}_i, \alpha_{n+i+1}, \dots, \alpha_{2n+1}) = \emptyset$.

$$\mathfrak{M}^+ \vDash \phi(\bar{a}) \Leftrightarrow \mathfrak{N} \vDash K_\phi(\bar{a}, \alpha). \quad (1)$$

The crucial point in this construction was the case $\phi(\bar{y}) = \exists x \psi(x, \bar{y})$. They proposed

$$K_{\exists x\psi(x,\bar{y})}(\bar{y}, \alpha) := \exists z_1 \exists z_2 \exists x (z_1 < \alpha < z_2 \wedge \forall z (z_1 < z < z_2 \rightarrow K_{\psi(x,\bar{y})}(x, \bar{y}, z)).$$

Since the 1-type $p \in S_1(M)$ is uniquely realizable, two convex to right and to left from α $M\alpha - 1$-formulas have solutions out of $p(\mathfrak{N})$. Thus for any $\bar{a} \in M$, if $\mathfrak{N} \vDash K_{\exists x\psi(x,\bar{y})}(\bar{a}, \alpha)$, then for some $b_1, b_2 \in M$,

$$\mathfrak{N} \vDash \exists x (b_1 < \alpha < b_2 \wedge \forall z (b_1 < z < b_2 \rightarrow K_{\psi(x,\bar{y})}(x, \bar{a}, z)).$$

This means that in an elementary submodel of $\mathfrak{N}$ the part of the last formula holds on $\mathfrak{M} \vDash \exists x \forall z (b_1 < z < b_2 \rightarrow K_{\psi(x,\bar{y})}(x, \bar{a}, z))$. Then there is an element $c \in M$ such that $\mathfrak{M} \vDash \forall z (b_1 < z < b_2 \rightarrow K_{\psi(x,\bar{y})}(c, \bar{a}, z))$. So, $K_{\psi(x,\bar{y})}(c, \bar{a}, z) \in p$.

Thus, any Σ^+-M-1-formula $\phi(x, \bar{a})$ has the set of its realizations, $\phi(\mathfrak{M}^+, \bar{a}) = K_\phi(\mathfrak{N}, \bar{a}) \cap M$, being a finite union of convex sets because $K_\phi(\mathfrak{N}, \bar{a})$ is a finite union of intervals and points. The elementary theory of $\mathfrak{M}^+$ is weakly o-minimal since the number of convex sets is bounded and consequently does not depend on parameters.

Approach of B.S. Baizhanov. For the case when $p \in S_1(M)$ is a non uniquely realizable type, B.S. Baizhanov proposed [2] (1995), on the base of theory of (non)orthogonality of 1-types and its classification made in [4], [5], [6], [8] (Pillay-Steinhorn, Marker, Mayer, Marker-Steinhorn, 1986–1994), to take the constants for $K_{\exists x\psi(x,\bar{y})}$ from an infinite indiscernible sequence $I = \langle \alpha_n \rangle_{n<\omega}$ over M and α_n from $p(\mathfrak{N})$. Taking into consideration that if $K_{\psi(x,\bar{y})}(\mathfrak{N}, \bar{a}, \bar{\alpha}_n) \cap M = \emptyset$, then there is a finite number irrational cuts (1-types over M) such that for any such 1-type $r \in S_1(M)$, $K_{\psi(x,\bar{y})}(\mathfrak{N}, \bar{a}, \bar{\alpha}_n)$ is a subset of

$$QV_r(\bar{\alpha}_n) := \{\beta \in r(\mathfrak{N}) \mid \text{there exists an } M\bar{\alpha}_n\text{-1-formula } \Theta(x, \bar{\alpha}_n), \text{ such that } \beta \in \Theta(\mathfrak{N}, \bar{\alpha}_n) \subset r(\mathfrak{N})\}.$$

The idea to use an indiscernible sequence consists from two parts.

B1. On the one hand, if for some $c \in M, \mathfrak{N} \vDash K_{\psi(x,\bar{y})}(c, \bar{a}, \bar{\alpha}_n)$, then for any $\bar{\gamma} = \langle \alpha_{i_0}, \dots, \alpha_{i_n} \rangle$ $(n < i_0 < \dots < i_n)$, $\mathfrak{N} \vDash K_{\psi(x,\bar{y})}(c, \bar{a}, \bar{\gamma})$, because $\bar{\alpha}_n$ and $\bar{\gamma}$ have the same type over over M.

B2. On the other hand, to find a sequence I such that for any $r \in S_1(M)$, for any $\bar{\gamma} = \langle \alpha_{i_0}, \dots, \alpha_{i_n} \rangle$ $(n < i_0 < \dots < i_n)$, $QV_r(\bar{\alpha}_n) \cap QV_r(\bar{\gamma}) = \emptyset$.

For find the indiscernible sequence I define the properties A1-A3 that follow from the classification of 1-types and theory non orthogonality of 1-types over sets in o-minimal theories.

A1. [5] (Marker 1986). *Let $q, r \in S_1(A)$, and let type $q(x) \cup r(y)$ be non complete (q is non weakly orthogonal to r, Shelah, 1978). Then there is an A-definable monotonic bijection $g: q(\mathfrak{N}) \rightarrow r(\mathfrak{N})$ and consequently, q is irrational if and only if r is irrational;*

q is uniquely realizable if and only if r is uniquely realizable.

Recall that if $q \in S_1(A)$ is irrational then $q(\mathfrak{N})$ is a convex non-definable set without maximal and minimal elements.

A2. *If $q \in S_1(A)$ is irrational, then for any $\bar{\gamma}$, $QV_q(\bar{\gamma}) = V_q(\gamma)$, here*

$$V_q(\bar{\gamma}) := \{\beta \in q(\mathfrak{N}) \mid \exists \delta_1, \delta_2 \in q(\mathfrak{N}), \text{ there exists an } A\bar{\gamma}\text{-1-formula } S(x, \bar{\gamma}), \delta_1 < S(\mathfrak{N}, \bar{\gamma}) < \delta_2, \beta \in S(\mathfrak{N}, \bar{\gamma})\}.$$

Indeed, the *quasi-neighborhood of* $\bar{\gamma}$ in q ($QV_q(\bar{\gamma})$) is the union of $A\bar{\gamma}$-definable sets, and any such definable set is a subset of $q(\mathfrak{N})$, a convex non-definable set without minimal and maximal elements. The last means that such a definable set is a subset of $V_q(\bar{\gamma})$ (*neighborhood of* $\bar{\gamma}$[n q). This explains equality of two convex sets.

A3. *If* $q \in S_1(A)$ *is irrational and non uniquely realizable, then for any* $\bar{\gamma} \in N$, if $QV_q(\bar{\gamma}) \neq \emptyset$ *then the 1-types* $q(x) \cup \{x < QV_q(\bar{\gamma})\}$ *and* $q(x) \cup \{QV_q(\bar{\gamma}) < x\}$ *are irrational and non uniquely realizable.*

By $A2$ and theorem of compactness there exist $\delta_1, \delta_2 \in q(\mathfrak{N})$ such that $\delta_1 < V_q(\bar{\gamma}) < \delta_2$ and because q is non uniquely realizable i.e. there is A-definable monotonic bijection $f: q(\mathfrak{N}) \to q(\mathfrak{N})$, $V_q(\mathfrak{N})$ can not have minimal and maximal element. Taking in consideration that for irrational q, $q(\mathfrak{N})$ is a convex non-definable set without maximal and minimal elements, $r_1 := tp(\delta_1|A\bar{\gamma})$ and $r_2 = tp(\delta_2|A\bar{\gamma})$ are irrational and since $f(V_q(\bar{\gamma})) = V_q(\bar{\gamma})$, f acts on $r_1(\mathfrak{N})$ and $r_2(\mathfrak{N})$. The last means r_1 and r_2 are non uniquely realizable.

Let $p_n(x) := p(x) \cup (QV_p(\bar{\alpha}_{n-1}) < x)$. Then by A2, A3 p_n is irrational, non uniquely realizable and finitely satisfiable in M since right sides of p and p_n coincide

For any $n, k, m < \omega$,

$$QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) < QV_{p_n}(\alpha_{n+k+1}, \alpha_{n+k+2}, \dots, \alpha_{n+k+m}), \tag{2}$$

and consequently, all these sets have empty intersection.

The proof of (2) is done by induction on m. Assume (2) for m. Denote by $r_1(x) = p_n \cup (x < QV_{p_n}(x)(\alpha_{n+k+1}, \dots, \alpha_{n+k+m}))$ and $r_2(y) = p_n(y) \cup (QV_{p_n}(\alpha_{n+k+1}, \dots, \alpha_{n+k+m}) < y)$, Suppose that

$QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) \cap QV_{p_n}(\alpha_{n+k+1}, \dots, \alpha_{n+k+m}, \alpha_{n+k+m+1}) \neq \emptyset$. Since the first set does not change, there exists $M\bar{\alpha}_n\alpha_{n+k+1} \dots \alpha_{n+k+m+1}$-formula $L(x, \alpha_{n+k+m+1})$ such $L(\mathfrak{N}, \alpha_{n+k+m+1}) \subset QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) \subset r(\mathfrak{N})$. Let β be end point of one of interval of formula L, then because p_n is non uniquely realizable then $\beta \in QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) \subset r(\mathfrak{N})$. Since $\beta \models r_1$ and $\alpha_{n+k+m+1} \models r_2$ by A1 there exists $M\bar{\alpha}_n\alpha_{n+k+1} \dots, \alpha_{n+k+m}$-definable monotonic function $f: r_2(\mathfrak{N}) \to r_1(\mathfrak{N})$ such that $f(\alpha_{n+k+m+1}) = \beta$. On other hand $\beta \in QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k})$ and consequently, there is $M\bar{\alpha}_{n+k} - 1$-formula $H(x)$ such that $\beta \in H(\mathfrak{N}) \subset QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) \subset r_1(\mathfrak{N})$. Then $\alpha_{n+k+m+1}$ belongs to $M\bar{\alpha}_{n+k+m}$-definable set $f^{-1}(H(\mathfrak{N})) \subset r_2(\mathfrak{N})$. This means $\alpha_{n+k+m+1} \in QV_p(\bar{\alpha}_{n+k+m})$. Contradiction.

It follow from (2) and A1 that for any $r \in S_1(M)$, if for any $i < n$, $p_i \perp^w r$ and $p_n \not\perp^w r$ then $QV_r(\bar{\alpha}) \cap QV_r(\bar{\alpha}_i, \alpha_{n+i+1}, \dots, \alpha_{2n+1}) = \emptyset$.

$$QV_r(\bar{\alpha}) \cap QV_r(\bar{\alpha}_i, \alpha_{n+i+1}, \dots, \alpha_{2n+1}) = \emptyset \tag{3}$$

Suppose for the formula $\psi(x, \bar{y})$ of signature Σ^+ corresponding formula of signature Σ is $K_{\psi(x,\bar{y})}(x, \bar{y}, \bar{\alpha}_n)$. Thus for any formula $K_\psi(x, \bar{a}, \bar{\alpha}_n)$ to have the solution in M it is sufficient to write the formula

$$K_{\exists x\psi(x,\bar{y})}(\bar{y}, \bar{\alpha}_{2n+1}) := \exists x(K_\psi(x, \bar{y}, \bar{\alpha}_n) \wedge$$
$$\wedge_{i \le n} K_\psi(x, \bar{y}, \bar{\alpha}_{n-i}, \alpha_{(n-i)+n+1}, \alpha_{(n-i)+n+2} \dots, \alpha_{2n+1})).$$

B.S. Baizhanov in 1996 obtained a classification of 1-types over a subset of a model of weakly o-minimal theory and solved the problem of expanding a model of weakly-o-minimal theory by a unary convex predicate in the preprint "Classifications of 1-types in weakly o-minimal theories and its applications" and submitted in the JSL, that revised version published in [9](2001). Ye.Baisalov and B. Poizat [10] (preprint 1996) in the paper on "beautiful" pairs of models of o-minimal theories proved the elimination of quantify $\exists x \in M$. It is difficult to say that the approach in [10] is alternative to approach elaborated in [2], because they used the same principles B1-B2 from [2].

We say that $\mathfrak{M}^+$ *the expansion by all externally definable subsets admits quantifier elimination*, if for any formula $\phi(\bar{y})$ of Σ^+ there exists Σ-formula $K_\phi(\bar{y}, \bar{z})$, there exists $\bar{\alpha} \in N \backslash M$ such that for any $\bar{a} \in M$ the following holds:

$$\mathfrak{M}^+ \models \phi(\bar{a}) \Leftrightarrow \models K_\phi(\bar{a}, \bar{\alpha}).$$

Approach of Shelah. In his paper, S. Shelah [11] (2004) considered a model of NIP theory and proved that the expansion by all externally definable subsets admits quantifier elimination and thereby is NIP. The key problem here is eliminating quantifier "there exists in the submodel". In his proof in the way of contradiction Shelah used an indiscernible sequence $\langle \bar{b}_n : n < \omega \rangle$ in order to show that if eliminating quantifier "there exists x in the submodel" $\varphi(x, \bar{a})$ fails, then $\varphi(\alpha, \bar{b}_n)$ holds iff n is even, for some α, which implies the independence property, for a contradiction.

V.V. Verbovskiy [12] (preprint 2005) found a somewhat simplified account of Shelah's proof, namely by using noting of a finitely realizable type. A. Pillay [13] (preprint 2006) gave two re-proofs of Shelah's theorem, the first going through quantifier-free heirs of quantifier-free types and the second through quantifier-free coheirs of quantifier-free types.

The analysis of approaches shows that the using the theory of orthogonality we can control the set of realizations of one-types. If we consider the complete theory satisfies A2, it gives the possibility to construct the indiscenible sequence satisfied the condition B2. Notice that the of indiscernible sequence constructed by mathematical induction satisfies the condition of finite realizability of an one-type of new element over model and beginning of sequence. The generalization of the approach for o-minimal model in case of non uniquely realizable one-type by introduction of generalization of the notions of (quasi)-neighborhood and almost (non)-orthogonality of two types gives the possibility to formulate the next

Theorem 1 Let T be a complete NSOP theory such that for any set A the following holds:

1) For any $p \in S_1(A)$, for any $\bar{\gamma}$, $QV_p(\bar{\gamma}) = V_p(\bar{\gamma})$

2) For any $p, q \in S_1(A)$ the following holds. If $p \not\perp^a q$, then $q \not\perp^a p$.

Then for model of the theory T the expansion of this model by one-type admits uniformly representation of Σ_p^+-formulas by Σ-formulas.

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НЕКОТОРЫЕ ВОПРОСЫ О ВНЕШНЕЙ ОПРЕДЕЛИМОСТИ

Аннотация. В статье рассматриваются различные подходы к концепции внешней определимости, разработанные в о-минимальных теориях. Пример о-минимальных теорий показывает, как внешние константы помогают определить существование решения в модели формулы с внешними константами. Сформулированы основные понятия, с помощью которых доказывается внешняя определимость. Дается краткий обзор результатов для зависимых теорий. В заключение сформулированы достаточные условия, так что теория обладает свойством внешней определимости. Дано краткое объяснение изложенной теоремы.

Ключевые слова: внешне определяемая, окрестность кортежа множества в типе, неортогональность двух типов

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СЫРТҚЫ ЕРЕЖЕЛЕРГЕ ҚАТЫСТЫ САУАЛДАР

Аннотация. Мақалада о-минималды теорияларда жасалған сыртқы анықтама тұжырымдамасына әртүрлі көзқарастар қарастырылған. О-минималды теориялардың мысалы сыртқы тұрақтылар формула моделінде ерітіндінің болуын анықтауға көмектесетінін көрсетеді. Негізгі ұғымдар сыртқы анықталуы дәлелденген, тұжырымдалған. Тәуелді теориялардың нәтижелеріне қысқаша шолу келтірілген.

Қорытындылай келе, теория сыртқы анықталу қасиетіне ие болатындай жеткілікті жағдайлар жасалған. Көрсетілген теоремаға қысқаша түсініктеме беріледі.

Түйін сөздер: сыртқы анықталған, типтегі жиынның көршілес, екі түрге жатпайтындығы.

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REFERENCES

[1] D. Macpherson, D. Marker, Ch. Steinhorn, Weakly o-minimal structures and real closed fields // Translations of the American Mathematical Society, 352 (2000), 5435–5483.

[2] B.S. Baizhanov, Extensions of o-minimal structures by convex unary predicates // In the book: "Research in the theory of algebraic systems" (editor T.A. Nurmagambetov), Karaganda State University, 1995, 6–24. (in Russian)

[3] A. Pillay, C. Steinhorn, Definable sets in ordered structures // Bulletin AMS, 11 (1984), 159–162.

[4] A. Pillay, C. Steinhorn, Definable sets in ordered structures I // Transactions of American Mathematical Society, 295 (1986), 565–592.

[5] D. Marker, Omitting types in o-minimal theories // The Journal of Symbolic Logic, 51 (1986), 63–74.

[6] L. Mayer, Vaught's conjecture for o-minimal theories // The Journal of Symbolic Logic, 53 (1988), 146–159.

[7] A. Pillay, Ch. Steinhorn, Definable sets in ordered structures III // Transactions of the American Mathematical Society 300 (1988), 469–476.

[8] D.Marker, Ch. Steinhorn, Definable types in o-minimal theories // The Journal of Symbolic Logic, 59 (1994), 185–198.

[9] B.S. Baizhanov, Expansion of a model of a weakly o-minimal theory by a family of unary predicates // The Journal of Symbolic Logic, 66 (2001), 1382–1414.

[10] Y. Baisalov, B. Poizat, Pairs of O-minimal structures // Journal of Symbolic Logic, 63:2 (1998), 570–578.

[11] S. Shelah, Dependent first order theories // Continued, Israel Journal of Mathematics 173: 1. (2009) https://doi.org/10.1007/s11856-009-0082-1

[12] V. Verbovskiy, Dependent theories. The Shelah's theorem // Proceeding of International conference "Contemporary problems of mathematics, informatics and control", dedicated to 60th anniversary of M.B. Aidarkhanov, October 2-3, 2008, Almaty, 439–441

[13] A. Pillay, On externally definable sets and a theorem of Shelah // Algebra, logic, set theory, Stud. Log.(Lond.), (2007) pdfs.semanticscholar.org

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